SEMI-ANNUAL STATUS REPORT

Volume II

RESEARCH IN AND APPLICATION OF MODERN AUTOMATIC CONTROL THEORY TO NUCLEAR ROCKET DYNAMICS AND CONTROL

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PREFACE

This report represents the completion of one phase of the study of the stability of control systems, a study sponsored by the National Aeronautics and Space Administration under Grant NsG-490 on research in and application of modern automatic control theory to nuclear rocket dynamics and control. The report is intended to be a self-contained unit and therefore repeats some of the work presented in previous status reports.

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ABSTRACT

In this work the Second Method of Liapunov, the Popov frequency criterion and the matrix-inequality method are used to study the stability of certain nonlinear and/or time-varying control systems. Systems with more than one nonlinear or time-varying element are considered, and the type of stability of interest is absolute stability; that is, global asymptotic stability.

The introductory material contains a description of the types of systems which are to be considered, stability definitions, and stability theorems of the Second Method. For systems with one nonlinear element, the Popov criterion and its geometric interpretation are given. The matrix-inequality method is used to develop the connection between the Second Method of Liapunov and the Popov criterion, thereby proving the Popov criterion. The Liapunov function used is of the Lurie type; that is, a quadratic form of the state variables plus the integral involving the nonlinearity.

Systems with a single time-varying element are considered next, and the use of a quadratic Liapunov function without the integral is shown to give results equivalent to those of Bongiorno, Sandberg and Narendra and Goldwyn. Inclusion of the integral of the time-varying elements results in a Liapunov function which is no longer $V(\underline{x})$, but is $V(\underline{x},t)$. The results of putting bounds on the integral of the time derivative of the nonlinearity, which appears in dV/dt, are easily

demonstrated by use of the matrix-inequality method. A lengthy example is used to indicate when this last criterion gives improved results.

The principal contribution of this work is an extension of the matrix-inequality method to systems with more than one nonlinear or time-varying element. The matrix-inequality method states that a scalar function of the real frequency must always be positive to conclude that a certain set of algebraic equations had a solution. The new result is that a matrix which is a function of real frequency must be positive definite for all real frequencies, if a correspondingly more involved set of algebraic equations is to have a solution. The new result allows stability criteria to be derived for systems with more than one nonlinear or time-varying element, which are analogous to the previous criteria for systems with one nonlinear or time-varying element.

Examples are included to illustrate the use of the new criteria, and a comparison with previous results is made. The case where the system equations contain a zero eigenvalue in their linear part is also discussed. An appendix is included in which the criteria described above is used to extend some results on stability of forced systems.

In conclusion, the main contribution of this work is an extension of the matrix-inequality method to systems with more than one nonlinearity. This results in new stability criteria which are extensions of criteria which exist for systems with a single nonlinear and/or time-varying element.

Chapter 1

INTRODUCTION TO THE PROBLEM

1.1 Introduction

The development of modern technology has brought forth many complicated devices and systems which defy analysis by the conventional methods of linear control system theory. Not only are there no design methods available for these systems, but even the problem of whether or not they are stable presents great difficulty. This work considers the stability of three types of systems which are described by ordinary differential equations: linear systems with time-varying parameters; nonlinear systems, especially those with more than one nonlinear element; and nonlinear, time-varying systems.

The tools used in studying these systems are the Second Method of Liapunov, the stability criterion which was developed by the Rumanian engineer V. M. Popov, and an approach to Popov's work known as the matrix-inequality method which was developed by the Russian mathematician, V. A. Yakubovich.

1.2 Historical Background

The trend in modern control theory has been away from the frequency domain, block diagram approach and toward what might be considered the "old-fashioned" differential equation representation of the control system. The main reasons for this are: first, nonlinear and/or time-varying systems cannot be handled by the frequency techniques that are so powerful for linear, non-time-varying systems; and second, the coming of age of the digital computer has enabled computations to be performed on large systems of differential equations.

One break in this return to the time domain has been in the area of stability theory, where the criterion derived by V. M. Popov (1961) has surprisingly put the study of the stability of a large class of nonlinear and time-varying systems back into the frequency domain. However, even here, the derivation starts out with the system represented by a set of differential equations rather than by a transfer function. For a system with one nonlinear or time-varying element, the transfer function of the linear, time-invariant part of the system is then used in obtaining a powerful geometric interpretation of the Popov criterion.

However, there is a direct connection between the work of Popov and the Second Method of Liapunov. By exploiting this connection the stability criteria developed in this work are derived. Liapunov developed his Direct or Second Method in the late nineteenth century, but it was not until the early 1940's in Russia and the early 1960's in the United States that engineers became interested in the theory. Popov developed his criterion for nonlinear systems in the late 1950's and early 1960's.

The two main additions to Popov's theory are the papers of Yakubovich (1962) and Kalman (1963). Their contributions are indicated at the appropriate place in the work which follows. The application of the Popov criterion to time-varying systems was first made by Rozenvasser (1963).

There is substantial literature on the subject of Liapunov's Second Method. The standard references are Liapunov's monograph (1949), the books of Hahn (1963), and LaSalle and Lefschetz (1961), and the article by Kalman and Bertram (1960). Besides the previously mentioned papers on the Popov criterion, there are the books by Aizerman and Gantmacher (1964) and Lefschetz (1965). The above mentioned books and papers contain extensive bibliographies.

Recently a great deal of work has been done on the problem of the stability of time-varying systems. Some work on time-varying circuits from the energy point of view was done by Darlington (1964), Rohrer (1964) and Kuh (1965). While of theoretical interest, these methods are not discussed here since other methods seem to give better results as far as stability is concerned. A real frequency criterion has been developed by Bongiorno (1963, 1964) for linear systems and Sandberg (1964) for nonlinear systems, both for the case of one time-varying element. Narendra and Goldwyn (1964) get similar results using the Second Method. These criteria will be shown to be equivalent to the Popov criterion.

There are also some theorems from the theory of linear differential equations with time-varying coefficients which seem to have been largely ignored in the engineering literature. A complete theory for linearly differential equations with periodic coefficients exists and is known as Floquet theory (Coddington and Levinson 1955). Parts of this theory have been used by various people in developing sufficient stability criterion for time-varying systems. However, the full use of the Floquet theory, which can easily be done using a digital computer, gives necessary and sufficient conditions for stability or, in other words, the exact stability information. There are also theorems available for linear systems with variations that go to zero as time increases. The theorems are due to Bellman (1953) and are discussed in Chapter 2.

1.3 Organization of the Work

The second chapter is essentially a background chapter in which the systems to be treated are described, and their stability is discussed. First there is a mathematical description of the general system under consideration, and then the various special cases of this general case are discussed. Then there are the definitions of the various types of stability which are needed and a discussion of absolute stability. Bellman's theorems on almost constant, linear, time-varying systems are then presented and their use discussed. Finally some theorems on the Second Method of Liapunov are given. There is a discussion of the difficulties involved in using the Second Method which indicates how the Popov criterion can help.

The third chapter introduces Popov's work for systems with one nonlinear element and includes a geometric interpretation of the results. There is a statement and proof of a lemma which is the heart of Yakubovich's matrix-inequality method. This is then used to show the connection between the Second Method and the Popov criterion.

Chapter 4 is devoted to systems with one time-varying element, either linear or nonlinear. The use of the Popov stability criterion for this type of system is shown to be valid, and the connection of the Popov criterion with the works of Bongiorno, Sandberg, and Narendra and

Goldwyn is given. This results in more geometric interpretations of the various criteria. The chapter concludes with an extension of the previous work due to Rekasius and Rowland (1965).

Chapter 5 contains most of the original contributions of this work. The extension of the previous work to systems with more than one nonlinear element is given. This requires extending the lemma of Chapter 3 from a scalar case to a matrix case. This is then used for time-varying systems to extend the result of Rekasius and Rowland to the case where more than one element is time-varying. There is also an extension of some work due to Letov (1961) in which he discusses the stability of systems with two actuators.

Chapter 6 contains conclusions and suggestions for further research. An appendix is included which presents an application of the previous work to forced systems, thereby extending the work of Yakubovich (1964) on nonlinear, forced systems.

1.4 Notation

Due to the large amount of different quantities involved in the mathematical derivations, it is necessary to use a mixture of the Greek and English alphabets. Capital English letters, such as A, B, P, and Greek letters with a bar over them, such as $\bar{\beta}$, are used to represent matrices. The exceptions to this are $V(\underline{x})$, which is used as the Liapunov function; $W(\underline{x})$ which is used in conjunction with $V(\underline{x}, t)$ in Chapter 2; and $G(s)$ or $G(j\omega)$ and $W(\omega)$, which are the transfer function and modified transfer function of the linear part of a nonlinear system with one nonlinear element. Small English letters which are underlined are vectors or column matrices, e.g., $\underline{c}$, $\underline{b}$, $\underline{x}$. Small Greek and English letters, subscripted or not, are scalars, such as, α, β_1, y_1, x.

The following are notations used in connection with matrix operations. A' is the transpose of the matrix A. A^* is the conjugate-transpose or adjoint of the matrix A. HeA is the Hermitian part of A and equals $\frac{1}{2}(A + A^*)$. The identity matrix is denoted by I. The notation $A > 0$ means that A is positive definite. Saying that A is a stable matrix means that all the eigenvalues of A are in the left half plane.

Chapter 2

SYSTEM REPRESENTATION AND STABILITY

2.1 Introduction

The purpose of this chapter is threefold. First, the classes of systems which are under consideration are discussed, then the stability of such systems is defined, and finally pertinent stability theorems are presented.

The second section begins with a discussion of the general nth order system with m nonlinear, time-varying elements. The various special cases of the general system are then given, that is, the linear, time-varying case, the single nonlinearity case, and the single nonlinearity with a zero eigenvalue in the linear part of the system.

In the third section the different kinds of stability which are needed are defined. Such stability concepts as global stability, asymptotic stability, uniform stability, and absolute stability are discussed. Then, in the fourth section, the stability of a special class of linear, time-varying systems known as "almost constant" systems is discussed. The theorems presented for this class of systems are due to Bellman (1953).

The last section presents stability theorems of the Second Method of Liapunov. The difficulties encountered in applying the Second Method are discussed, especially in regard to time-varying systems. This leads into the Popov criterion which is presented in Chapter 3.

2.2 System Representation

Many complex systems, such as systems with many control actuators, can be described by a matrix set of equations of the form

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{u} &= -\underline{f}(\underline{\sigma},t), \quad f(\underline{0},t) = \underline{0} \qquad (2\text{-}1) \\ \underline{\sigma} &= C'\underline{x} \end{aligned}$$

where A is a constant, n by n matrix with all its eigenvalues in the left-half plane (such a matrix is referred to as a stable matrix), B and C are n by m matrices, $\underline{x}$ is a n-dimensional state vector, $\underline{u}$ is the m-dimensional control vector, and

$$\underline{f}(\underline{\sigma},t) = \begin{bmatrix} f_1(\sigma_1,t) \\ f_2(\sigma_2,t) \\ \cdot \\ \cdot \\ \cdot \\ f_m(\sigma_m,t) \end{bmatrix}$$

where $0 \leq f_i(\sigma_i,t)/\sigma_i \leq k_i$. It is assumed that the rank of B is equal to the dimension of the control vector $\underline{u}$. If this is not true, then one can always reduce the number of control variables by means of the proper linear transformations until it is true (Melsa 1965, ch. 7).

The restriction of the $f_i(\sigma_i,t)$ to the sector $[0, k_i]$ is no real restriction since any system of the same type, but with some nonlinearity $g_i(\sigma_i,t)$ such that $k_{ai} \leq g_i(\sigma_i,t)/\sigma_i \leq k_{bi}$, can be put into the proper form by the substitution $f_i(\sigma_i,t) = g_i(\sigma_i,t) - k_{ai}\sigma_i$. As usual the square brackets are used to indicate the closed interval. The half-open interval $0 < f(\sigma_i)/\sigma_i \leq k_i$ is indicated by $(0, k_i]$.

A special case of the general system is the linear, time-varying case. The equations are the same except that now the vector $\underline{u}$ is given by

$$\underline{u} = -F(t)\underline{\sigma},$$

where $F(t) = \text{diag}(f_1(t), \ldots, f_m(t))$, $0 \leq f_i(t) \leq k_i$. Putting the expression for $\underline{\sigma}$ into this equation and then putting $\underline{u}$ into the differential equation results in the linear, time-varying matrix equation

$$\dot{\underline{x}} = (A - B\,F(t)\,C') \,\underline{x} \tag{2-2}$$

The stability of this equation in the special case that F(t) approaches the zero matrix as t approaches infinity is discussed in section five of this chapter.

Another very important case of the general system (2-1) is the case where m = 1, that is, the single nonlinear and/or time-varying element system. Repeating the system equations for this case gives

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ u &= -f(\sigma,t), \quad 0 \leq f(\sigma,t)/\sigma \leq k \\ \sigma &= \underline{c}'\underline{x} \qquad\qquad f(0,t) = 0 \end{aligned} \tag{2-3}$$

This is just the differential equation of the familiar, single-loop, control system, Fig. 1. The transfer function of this system in terms of the above matrices can be calculated as follows. Take the Laplace transform of the differential equation in (2-3). The result is

$$s\underline{x}(s) = A\,\underline{x}(s) + \underline{b}u(s)$$

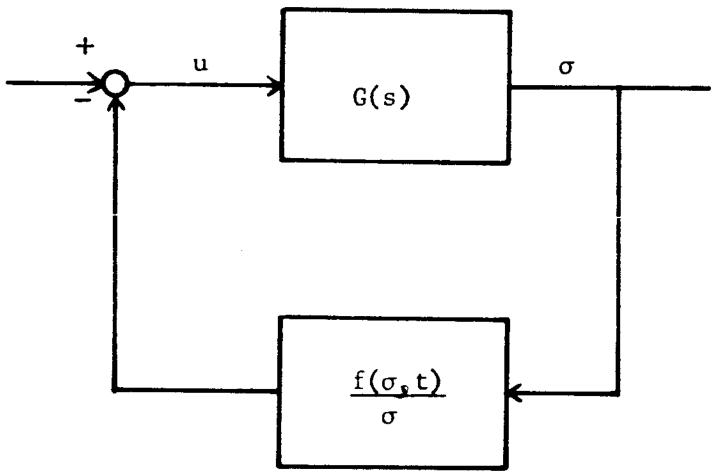


Fig. 1. n-th Order System with One Nonlinearity.

σ and u Variables Defined.

where the initial conditions are assumed to be zero as usual. Solve this equation for $\underline{x}(s)$.

$$(sI - A)\,\underline{x}(s) = \underline{b}\,u(s)$$

$$\underline{x}(s) = (sI - A)^{-1}\,\underline{b}\,u(s)$$

Substitute this equation into the equation for $\sigma(s)$. The result is

$$\sigma(s) = \underline{c}'\underline{x}(s) = \underline{c}'(sI - A)^{-1}\,\underline{b}\,u(s)$$

The transfer function is given by the ratio of $\sigma(s)/u(s)$.

$$\frac{\sigma(s)}{u(s)} = G(s) = \underline{c}'(sI - A)^{-1}\,\underline{b} \tag{2-4}$$

The block diagram of this system is given in Fig. 1.

Another important class of systems with one nonlinearity is the case of systems with a pure integration in the open loop system. The equations are similar to (2-3) except that the A matrix has one zero eigenvalue and all its other eigenvalues are in the left-half plane. When this is true, the dimension of the A matrix can be reduced by one by means of suitable linear transformations, and the zero eigenvalue equation is removed from the set of equations given in matrix form. The resulting set of equations is

$$\begin{aligned} \dot{\underline{y}} &= A_1\,\underline{y} + \underline{b}_1 u \\ u &= -f(\sigma) \\ \dot{\xi} &= f(\sigma) \\ \sigma &= \underline{c}_1'\,\underline{y} - \gamma\xi \end{aligned} \tag{2-5}$$

Again the transfer function of this set of equations can be found and it is

$$G(s) = \underline{c}_1' (sI - A_1)^{-1} \underline{b}_1 + \frac{\gamma}{s} \qquad (2\text{-}6)$$

or $$G(s) = G_1(s) + \frac{\gamma}{s}$$

One possible block diagram of this transfer function is given in Fig. 2. If the block diagram is given and the system under consideration has a pole at the origin, the proper state variables for writing the differential equation in the form of (2-5) can be obtained by first breaking up the block diagram as shown in Fig. 2. It can be seen from this that the quantity γ is actually the gain constant of the system and therefore must be positive.

In identifying the types of systems in the single nonlinearity case, the terminology of Aizerman and Gantmacher (1964) is followed, and the case of the A matrix with no zero or pure imaginary eigenvalues is called the principal case. Other cases are called particular cases, and, when the A matrix has one zero and no imaginary eigenvalues, it is called the simplest particular case. Also, in the particular case the nonlinear sector must be $0 < f(\sigma,t)/\sigma \leq k$, that is, $f(\sigma)/\sigma$ is not allowed to be zero. If $f(\sigma)/\sigma$ is allowed to be zero, then the integration term of the system would just integrate without any feedback and the system would be unstable.

2.3 Stability Definitions

It is assumed that the only singular point of (2-1) is the origin so that $\underline{x} = \underline{0}$ is a point solution of the equation. Then the

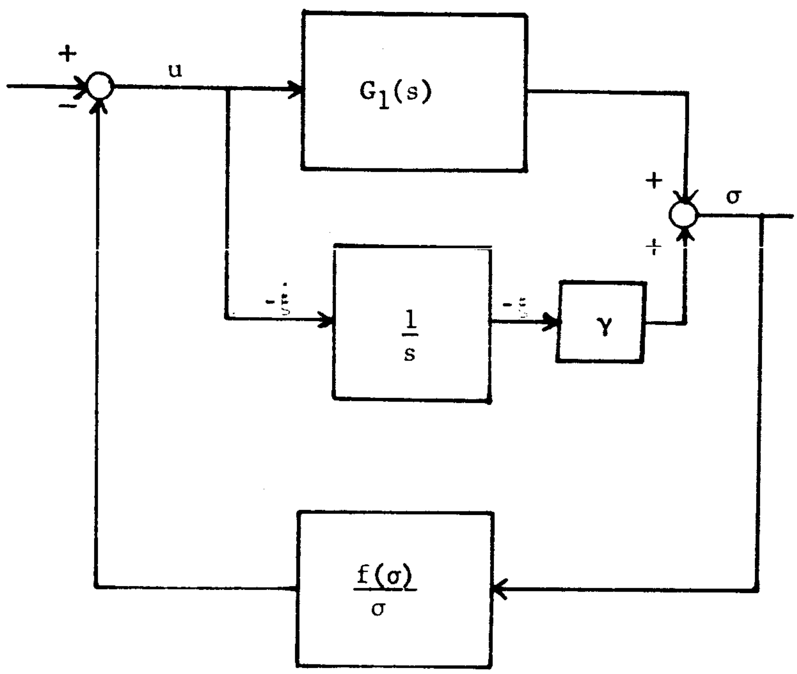


Fig. 2. Block Diagram Defining Variables for the Simplest Particular Case.

stability of (2-1) is defined as the stability of the solution $\underline{x} = \underline{0}$.

Definition 2-1: The null solution of the system (2-1) is said to be Liapunov stable (or just stable) at $t = t_o$, provided that for an arbitrary positive $\epsilon > 0$ there is a $\delta = \delta(\epsilon, t_o)$ such that whenever $|\underline{x}(t_o)| < \delta$, the inequality $|\underline{x}(\underline{x}(t_o), t_o, t)| < \epsilon$ is satisfied for all $t \geq t_o$.

Definition 2-2: The null solution of the system (2-1) is said to be asymptotically stable if it is stable, and if the limit as t approaches infinity of $\underline{x}(\underline{x}(t_o), t_o, t)$ equals zero.

Some other stability concepts are also needed. If the quantity δ appearing in Definition 2-1 does not depend on t_o, then the system is uniformly stable. If the system is asymptotically stable for all $\underline{x}(t_o)$ in the entire state space, then the system is globally, asymptotically stable. Kalman and Bertram (1960) give the following as the conditions for uniform, global, asymptotic stability

Definition 2-3: The equilibrium state $\underline{x} = \underline{0}$ is globally, uniformly asymptotically stable if

(a) it is uniformly stable

(b) it is uniformly bounded, i.e., given any $\epsilon > 0$ there is some $\delta(\cdot)$ such that $|\underline{x}(t_o)| \leq \epsilon$ implies $|\underline{x}(\underline{x}(t_o), t_o, t)| \leq \delta$ for all $t \geq t_o$.

(c) every motion converges to $\underline{x} = \underline{0}$ as t approaches infinity uniformly in t_o and $|\underline{x}(t_o)| \leq \epsilon$, when ϵ is fixed but arbitrarily large; i.e., given any $\epsilon > 0$ and $\delta > 0$ there is some $T(\delta, \epsilon)$ such that $|\underline{x}(t_o)| \leq \epsilon$ implies $|\underline{x}(\underline{x}(t_o), t_o, t)| \leq \delta$ for all $t \geq t_o + T$.

Also, if the system is globally, asymptotically stable for any $f_i(\sigma_i)$ in the sector $[0,k_i]$, then the system is absolutely stable. The absolute stability of the time-varying system is defined as uniform, global asymptotic stability for any $f_i(\sigma_i,t)$ in the sector $[0,k_i]$. In what is to follow the concept of absolutely stable systems plays a large part.

There have been some objections to trying to find absolute stability. It can be said that one is not really interested in absolute stability since systems don't operate in the entire state space, so that better results should be forthcoming if an operating region about the origin is considered, and then asymptotic stability is shown in that region. An answer to this is the fact that just because absolute stability can be shown for a given differential equation does not mean that the system that the equation represents is absolutely stable. It is the differential equation which is chosen to model a given system which is only valid in some region of the state space, and not the stability properties of that equation.

One other objection is that absolute stability puts no restriction on the slopes of the nonlinearity, as all that is required is that it remain in the sector. If the slope of the nonlinearity is restricted, perhaps some better answers would result in many cases. This is actually a current research area with the results of Brockett and Willems (1965) being about the only indication of success in this area.

Before the stability theorems of the Second Method of Liapunov are given, some stability theorems of a special class of linear time varying systems, called "almost constant" systems, are discussed. These theorems have largely been ignored in the engineering literature and are included for completeness.

2.4 The Stability of Almost Constant Systems

In his book on the stability theory of differential equations, Bellman (1953) presents some theorems on a class of linear, time-varying systems which he calls "almost constant". The system is represented by the equation $\dot{\underline{x}} = A(t)\,\underline{x}$, where the terminology "almost constant" comes from the condition that the limit as t approaches infinity of A(t) equals a constant matrix A. This equation can be a special case of (2-2). Writing (2-2) as

$$\dot{\underline{x}} = (A + B(t))\underline{x} \tag{2-7}$$

where

$$\lim_{t \to \infty} B(t) = 0$$

puts the equation in the proper form to apply the theorems.

Two theorems for this type of system are now stated without proof; the proof is in Bellman[9]s book.

Theorem 2-1: If all solutions of $\dot{\underline{y}} = A\underline{y}$, where A is a constant matrix, are bounded as t approaches infinity, the same is true of the solutions of (2-7) provided that

$$\int_{t_0}^{\infty} |B(t)| \quad dt < \infty$$

Theorem 2-2: If all solutions of $\dot{\underline{y}} = A\underline{y}$ approach zero as t approaches infinity, the same is true for the solutions of (2-7) provided that $|B(t)| \leq c$ for $t \geq t_0$, where c is a constant which depends on A.

Although these theorems deal with a large class of linear systems, they may lead to trouble in engineering work unless care is taken in their application. The problem is that although the linear system is eventually stable, it may have solutions which grow to very large values before finally approaching zero. When this is the case, the linear model which gives the equations may no longer be valid, and the physical system could be unstable.

If the linear model is no longer valid, then Theorems 2-1 and 2-2 are not valid, and a nonlinear model of the system along with the stability theorems of the Second Method of Liapunov must be used. These stability theorems are presented next.

2.5 Stability Theorems of the Second Method of Liapunov

Before stating some stability theorems of the Second Method of Liapunov, a few definitions need to be made. These definitions concern real, scalar functions of the state $\underline{x}$ and the time t.

Definition 2-4: A real scalar function $V(\underline{x})$ is called positive definite (positive semidefinite) if in a neighborhood of the origin $V(\underline{x}) > 0$ ($V(\underline{x}) \geq 0$) and $V(\underline{0}) = 0$.

Definition 2-5: A real scalar function $V(\underline{x},t)$ is called positive definite in a region of the origin if

$$V(\underline{x},t) \geq W_1(\underline{x})$$

and

$$V(\underline{0},t) = 0$$

where $W_1(\underline{x})$ is positive definite.

Definition 2-6: A real scalar function $V(\underline{x},t)$ is called negative definite if $-V(\underline{x},t)$ is positive definite.

Using these definitions the following theorems (Szego, 1961) can now be stated

Theorem 2-3: If for $t \geq t_o$ there exists a real scalar function $V(\underline{x},t)$ in the neightborhood S of the origin, $V(\underline{x},t)$ being continuous and possessing continuous first partial derivatives with respect to x_i and t, and satisfying

1) $V(\underline{x},t)$ is positive definite in S for $t \geq t_o$
2) $\dot{V}$ is not positive (i.e., negative semi-definite) in S for $t \geq t_o$

then the trivial solution $\underline{x} = \underline{0}$ of (2-1) is stable (Liapunov stable).

Theorem 2-4: If conditions 1) and 2) of the above theorem are changed to

3) $V(\underline{x},t)$ is positive definite and also dominated for $t \geq t_o$ by another $W_2(\underline{x})$ (i.e., $W_1(\underline{x}) \leq V(\underline{x},t) \leq W_2(\underline{x})$)

4) $\dot{V}$ is negative definite in S for $t \geq t_o$

then the trivial solution $\underline{x} = \underline{0}$ of (2-1) is asymptotically stable.

Theorem 2-5: If in the above two theorems S is the entire state space, and in addition

$$\lim_{\underline{x} \to \infty} V(\underline{x},t) = \infty$$

uniformly on t, $t \geq t_o$, **then the trivial solution is, respectively, globally, uniformly stable and globally, uniformly, asymptotically stable.**

For the case of time invariant systems there is an extension of the above theorem which states that asymptotic stability can be concluded for $\dot{V}(\underline{x}) \leq 0$ provided that $\dot{V}(\underline{x})$ is not identically zero for any solution other than $\underline{x} = 0$. One of the problems with time-varying systems is the fact that this extension is not valid, therefore requiring a definite $\dot{V}$.

One of the main factors holding back the application of the Second Method is the lack of methods for finding the best V-function for a given system. This is especially true for the case of time-varying systems. There does not seem to be any method available which can be used to generate Liapunov functions which have an explicit dependence on time. Therefore, either V-functions are generated which have no dependence on time or specific V-functions are picked.

One class of V-functions which has received much attention is the Lurie type, a positive definite quadratic form of the state variables plus integrals of the nonlinear terms. The V-functions are positive definite in the entire state space and $\dot{V}$ can be put in a form such that, if certain conditions are satisfied, it must be negative definite, and absolute stability is concluded. For systems with one nonlinear element these certain conditions are the Popov frequency criterion. For the case of the single nonlinearity system the Popov criterion gives necessary and sufficient conditions for the existence of this type of V-function. The fact that Popov's condition gives necessary and sufficient conditions for the existence of the V-function of the Lurie type in the simplest particular case has been known since Yakubovich's work (1962). However, recently Yakubovich (1964b) has shown that this is true for the principal case also.

As will be seen in Chapter 3, the advantage of the Popov criterion over the straightforward application of the Second Method is the ease with which it is used. Another advantage is that since it is a necessary and sufficient condition for the existence of a V-function of the proper form, it actually gives the results which are equivalent to finding the best V-function of that specific type (Aizerman and Gantmacher 1964, Appendix). This is important in time-varying systems, as can be seen by considering just the simple quadratic form of the state variables as the V-function. The use of the Popov criterion gives the best quadratic $V(\underline{x})$ for a given system. If the Popov criterion did not exist, this best $V(\underline{x})$ could only be found by a long and complicated search procedure, since $\dot{V}$ is a function of the system parameters as well as the particular quadratic form chosen for $V(\underline{x})$. For high order systems a considerable amount of work is required to do this.

Another advantage of using the Popov criterion in conjunction with the Lurie type V-function is that V and $\dot{V}$ do not have to be tested for definiteness, as the satisfaction of the Popov criterion guarantees that V is positive definite and $\dot{V}$ is negative definite. In trying to find Liapunov functions for high order systems by other techniques, one of the main difficulties is that there is no easy way of testing high order non-quadratic functions for definiteness.

Chapter 3 presents an extensive discussion of the Popov criterion and its relationship with Lurie type V-functions. The purpose of Chapter 3 is to develop a background from which the stability of system (2-1) can be studied.

Chapter 3

THE STABILITY CRITERION OF POPOV

3.1 Introduction

In this chapter the stability criterion which was formulated by the Rumanian engineer V. M. Popov is presented. Popov's work is concerned mainly with the absolute stability of the single, time-invariant, nonlinearity type of system given by (2-3). Only this case is discussed in this chapter with extensions to the time varying case to appear in the next chapter.

There has been much research in the last fifteen years on the absolute stability problem. This research was initiated by the Russian Lurie, and it concerns finding sufficient conditions for the stability of (2-3) by using a Liapunov function which is a quadratic form of all the state variables plus an integral of the nonlinearity. The type of Liapunov function is sometimes referred to as the Lurie type.

In the late 1950's Popov began working on frequency domain criteria for nonlinear systems. He published his main paper in 1961, and, in a short time, Yakubovich (1962,1964b) and Kalman (1963) completed Popov's work in an important way. The result is that the Popov criterion gives necessary and sufficient conditions for the existence of a Liapunov function V of the Lurie type, which insures the absolute stability of the system.

The main advantage of the Popov criterion over the Liapunov function method is that it can be interpreted graphically in a manner which requires just the polar plot of the amplitude and phase of a modified frequency function. This function is obtained by slightly modifying the system transfer function. Therefore, the differential equations of the system do not actually have to be known, and, what is even more important, high order systems can be handled as easily as low order.

The chapter has two main sections. Section 3.2 contains a statement of the Popov stability criterion and develops its geometric interpretation. In the last section the relationship between the Popov criterion and the Second Method of Liapunov is given. This is done by means of a lemma which is basically the matrix-inequality method of Yakubovich. Both the principal case (2-3) and the simplest particular case (2-5) are considered.

3.2 The Popov Criterion

In this section the Popov stability criterion is stated, and a geometric interpretation of it is given. The systems considered are the principal case and the simplest particular case of the class of systems with one time-invariant nonlinearity. For convenience the equations of these systems are repeated. The principal case is

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ u &= -f(\sigma), \quad 0 \leq f(\sigma)/\sigma \leq k \qquad\qquad (2\text{-}3) \\ \sigma &= \underline{c}'\underline{x} \end{aligned}$$

The block diagram is given in Fig. 1 where

$$G(s) = \underline{c}'(sI - A)^{-1}\,\underline{b} = \underline{c}'A_s^{-1}\,\underline{b} \tag{2-4}$$

The equations of the simplest particular case are

$$\begin{aligned} \dot{\underline{y}} &= A_1\underline{y} + \underline{b}_1 u \\ u &= -f(\sigma) \quad 0 < f(\sigma)/ \leq k \\ \dot{\xi} &= f(\sigma) \\ \sigma &= \underline{c}_1'\underline{y} - \gamma\xi \\ G(s) &= \underline{c}_1'(sI - A_1)^{-1}\underline{b}_1 + \frac{\gamma}{s} \end{aligned} \tag{2-5}$$

For the particular cases it is necessary to introduce the concept of stability-in-the-limit. The particular case is said to be stable in the limit if for $u = -\epsilon\sigma$, with $\epsilon > 0$ and sufficiently small, the linear system obtained from (2-5) is asymptotically stable. This is to make sure that the root locus of the linear system is in the left half plane for all linear gain between zero and k. This is a more significant problem for systems where the A matrix has a double root at the origin or pure imaginary zeros. These cases are not considered here. The reason for having $f(\sigma)/\sigma$ greater than zero, rather than greater than or equal to zero, for the simplest particular case is discussed in section 2.2.

Now that the class of systems has been specified, the V. M. Popov stability criterion can be stated. The statement of the following theorem is essentially the same as that given in Aizerman and Gantmacher (1964).

Theorem 3-1: For the principal case of (2-3) to be absolutely stable in the sector $[o, k]$ and for the simplest particular case (2-5) to be absolutely stable in the sector $(o, k]$, it is sufficient that there exist a finite real number β such that for all real $\omega \geq 0$ the following inequality is satisfied.

$$\mathrm{Re}(1 + j\beta\omega)G(j\omega) + \frac{1}{k} > 0 \tag{3-1}$$

The importance of the Popov criterion is due in a large part to its simple geometric interpretation. A new function $W(\omega)$, called the modified frequency function, is defined such that

$$\begin{aligned} \mathrm{Re}W(\omega) &= \mathrm{Re}G(j\omega) \\ \mathrm{Im}W(\omega) &= \omega \mathrm{Im}G(j\omega) \end{aligned} \tag{3-2}$$

so that the polar plot of $W(\omega)$ is obtained from that of $G(j\omega)$ by multiplying all its ordinates by the corresponding value of ω. Here it is assumed that the transfer function $G(s)$ always has more poles than zeros, so that $\lim_{\omega \to \infty} G(j\omega) = 0$. However, if there is only one more pole than zero, then $\lim_{\omega \to \infty} W(\omega)$ has a limit point on the imaginary axis not at the origin.

The modified frequency response function is used to obtain the geometric interpretation. Let

$$W(\omega) = X + jY$$

then

$$\mathrm{Re}(1 + j\beta\omega)G(j\omega) = \mathrm{Re}G(j\omega) - \beta\omega \mathrm{Im}G(j\omega) = X - \beta Y$$

Hence (3-1) can be written as

$$X - \beta Y + \frac{1}{k} > 0 \text{ for all } \omega \geq 0 \tag{3-3}$$

The equation

$$X - \beta Y + \frac{1}{k} = 0 \qquad (3\text{-}4)$$

is the equation of a straight line with slope $1/\beta$, which passes through the point $-1/k$ on the real axis. As in Aizerman and Gantmacher, this line is called the Popov line. The inequality (3-3) is valid if the modified frequency plot is in that part of the plane which is to the right of the $-1/k$ point and does not intersect the Popov line. Figure 3 shows two possible stable systems.

In the case $\beta = 0$, the modified frequency response does not have to be used. In that case (3-1) reduces to

$$\mathrm{Re}G(j\omega) + \frac{1}{k} > 0 \qquad (3\text{-}5)$$

so that as long as the plot of $G(j\omega)$ is to the right of the vertical line through $1/k$ (i.e., the slope $1/\beta$ is infinite), the system (2-3) or (2-5) is absolutely stable. An example of this is shown in Fig. 4.

A more complete look at the various results which are available for the different particular cases of the system (2-3) is given in Aizerman and Gantmacher (1964) and in the series of papers by Yakubovich (1963a, 1963b, 1964a).

3.3 The Relation Between the Popov Criterion and the Second Method

In this section the relation of the Popov criterion (3-1) to a Lurie type Liapunov function is discussed. Two Liapunov functions are used; V_0 for the principal case and V_1 for the simplest particular case.

$$V_0 = \underline{x}'P\underline{x} + \beta \int_0^{\sigma} f(z)dz \qquad (3\text{-}6)$$

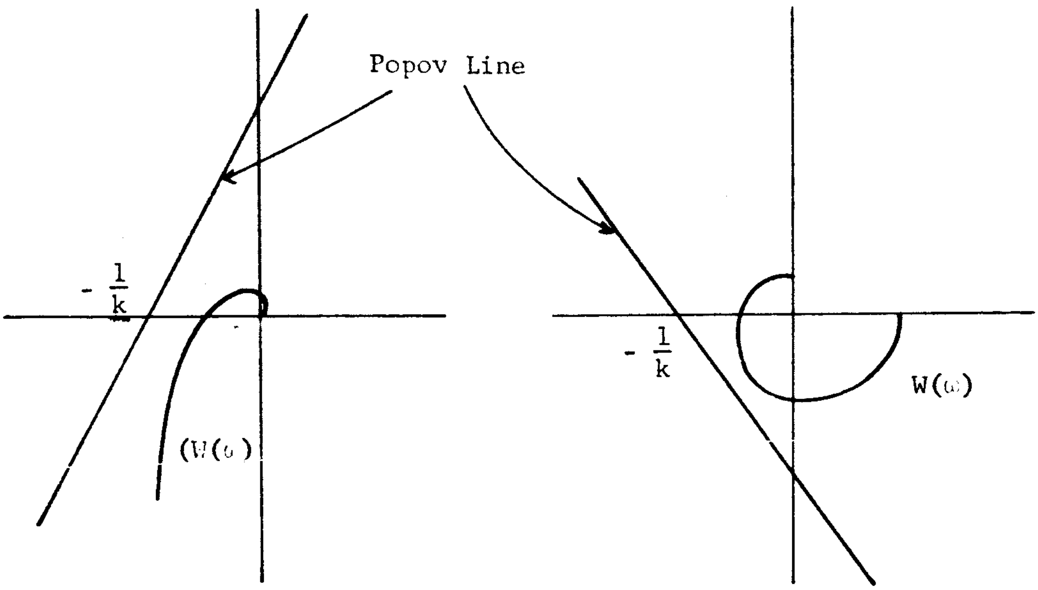


Fig. 3. Geometric Interpretation of the Popov Criterion - Stable Systems.

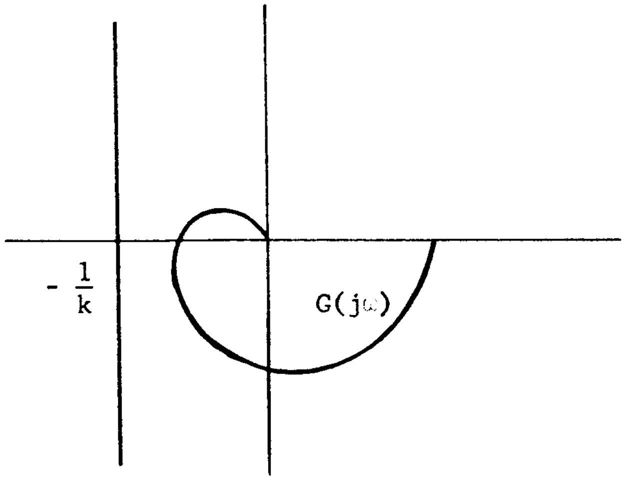


Fig. 4. Popov Criterion - $\beta = 0$ - Stable System

$$V_1 = \underline{x}'P\underline{x} + \alpha(\sigma - \underline{c}'\underline{x})^2 + \beta \int_0^{\sigma} f(z)dz \qquad (3\text{-}7)$$

Here P is a positive definite, symmetric matrix. Popov also considered adding the term $\underline{r}'\underline{x}$ to V_1 to get the most general quadratic form, but he proved that it is necessary that $\underline{r} = \underline{0}$. In Fig. 3 the geometric criterion is shown for β both positive and negative. Aizerman and Gantmacher (1964, p. 58) show that these two different cases are related by a linear change of variables which just interchanges the sides of the nonlinear sector $[0, k]$. For this reason only the case of $\beta \geq 0$ is considered in what follows.

Before proceeding any further, a lemma is proven which is of great use in what is to follow. This lemma is due originally to Yakubovich (1962) with the sufficiency proof following Lefschetz's (1965) version of Kalman's (1963) work. The term ρR does not appear in the above works but is inspired by the work of Rekasius and Rowland (1965) whose results are stated as a corollary. In most applications the term $\rho R = 0$, but since it is needed for some special cases, it is included in the derivation which follows.

Define A_ω by

$$A_\omega = (j\omega I - A) \qquad (3\text{-}8)$$

Since the matrix A has all its eigenvalues in the left half plane, the matrix A_ω is always nonsingular for all ω and A_ω^{-1} exists.

Lemma 1: Given the stable, n by n, real, matrix A; symmetric, n by n, real, matrices $D > 0$ and $R \geq 0$; n-vectors $\underline{g} \neq 0$ and $\underline{h} \neq 0$; and scalars $\tau \geq 0$, $\epsilon > 0$, and ρ such that the right hand side of (a) is negative definite; then a necessary and sufficient condition for the existence of a solution as a symmetric, n by n, real, matrix P (necessarily > 0) and an n-vector $\underline{q}$ of the system

$$A'P + PA = -\underline{q}\,\underline{q}' - \rho R - \epsilon D \quad \text{(a)}$$

$$P\underline{g} - \underline{h} = \sqrt{\tau}\;\underline{q} \quad \text{(b)}$$

is that ϵ be small enough, and that the relation

$$\tau + 2\mathrm{Re}\,\underline{h}'A_\omega^{-1}\,\underline{g} - \rho\underline{g}A_\omega^{-1*}\,RA_\omega^{-1}\,\underline{g} > 0 \qquad (*)$$

be satisfied for all real ω .

Proof of Necessity

The identity

$$A'P + PA = -(PA_\omega + A_\omega^{*}P) \qquad (3\text{-}9)$$

is needed first. This is obtained by adding and subtracting $j\omega P$ to $A'P + PA$.

$$\begin{aligned} A'P + PA &= A'P + j\omega P + PA - j\omega P \\ &= -(-j\omega I - A')P - P(j\omega I - A) \\ &= -A_\omega^{*}P - PA_\omega \end{aligned}$$

The identity (3-9) is used in (a) to get

$$PA_\omega + A_\omega^{*}P = \underline{q}\,\underline{q}' + \rho R + \epsilon D \qquad (3\text{-}10)$$

This is premultiplied by $\underline{g}'A_\omega^{*\,-1}$ and postmultiplied by $A_\omega^{-1}\underline{g}$ giving

$$\underline{g}'PA_\omega^{-1}\,\underline{g} + \underline{g}'A_\omega^{*-1}P\underline{g} = \underline{g}'A_\omega^{*-1}\underline{q}\,\underline{q}'A_\omega^{-1}\,\underline{g}$$

$$+ \rho\underline{g}'A_\omega^{*-1}\,RA_\omega^{-1}\,\underline{g} + \epsilon\underline{g}'A_\omega^{*-1}DA_\omega^{-1}\,\underline{g} \qquad (3\text{-}11)$$

Then by using (b) $P\underline{g}$ is replaced by $\sqrt{\tau}\;\underline{q} + \underline{h}$ giving

$$\sqrt{\tau}\, \underline{q}' A_\omega^{-1} \underline{g} + \underline{h}' A_\omega^{-1} \underline{g} + \underline{g}' A_\omega^{*-1} \sqrt{\tau}\, \underline{q} + \underline{g}' A_\omega^{*-1} \underline{h}$$

$$= \underline{g}' A_\omega^{*-1} \underline{q}\, \underline{q}' A_\omega^{-1} \underline{g} + \rho\, \underline{g}' A_\omega^{*-1} R A_\omega^{-1} \underline{g} + \epsilon \underline{g} A_\omega^{*-1} D A_\omega^{-1} \underline{g}$$

(3-12)

Since $\underline{h}' A_\omega^{-1} \underline{g} + \underline{g}' A_\omega^{*-1} \underline{h} = 2 \operatorname{Re} \underline{h}' A_\omega^{-1} \underline{g}$, then, by rearranging the terms, (3-12) becomes

$$2\operatorname{Re}\underline{h}' A_\omega^{-1} \underline{g} - \rho\, \underline{g}' A_\omega^{*-1} R A_\omega^{-1} \underline{g}$$

$$= \underline{q}' A_\omega^{-1} \underline{g}^2 - 2\sqrt{\tau} \operatorname{Re}\underline{q}' A_\omega^{-1} \underline{g} + \varepsilon \tag{3-13}$$

where $\varepsilon = \epsilon\, \underline{g}' A_\omega^{*-1} D A_\omega^{-1} \underline{g} > 0$. That ε is greater than zero can be seen by considering $D > 0$ as a Hermitian **matrix**. The matrix $D_1 = A_\omega^{*-1} D A_\omega^{-1}$ is the Hermitian matrix deduced from D by the change of coordinates $\underline{y} = A_\omega^{-1} \underline{x}$. Hence, $D_1 > 0$ and $\underline{g}' D_1 \underline{g} = \underline{g}' A_\omega^{*-1} D A_\omega^{-1} \underline{g} > 0$. Adding τ to both sides of (3-13) gives

$$\tau + 2\operatorname{Re}\underline{h}' A_\omega^{-1} \underline{g} - \rho\, \underline{g}' A_\omega^{*-1} R A_\omega^{-1} \underline{g}$$

$$= \left| \underline{q}' A_\omega^{-1} \underline{g} - \sqrt{\tau} \right|^2 + \varepsilon \tag{3-14}$$

Since the right side of (3-14) is always positive, the result is

$$\tau + 2\operatorname{Re}\underline{h}' A_\omega^{-1} \underline{g} - \rho\, \underline{g}' A_\omega^{*-1} R A_\omega^{-1} \underline{g} > 0 \tag{*}$$

Therefore, starting with (a) and (b) and assuming that all the quantities exist, it has been shown that (*) is necessarily true.

Before going on to the sufficiency proof an additional observation, due to Kalman (1963), is made. When the pair $(A, \underline{g})$ is completely controllable, i.e., $\det(\underline{g}, A\underline{g}, \ldots, A^{n-1}\underline{g}) \neq 0$, the matrix A can be represented in phase variables with $\underline{g}' = (0\ 0\ \ldots\ 0\ 1)$. Let $A_s = (sI - A)$. Then, for the given choice of A and $\underline{g}$ the expression $\underline{h}'A_s^{-1}\underline{g}$ is written as

$$\underline{h}'A_s^{-1}\ \underline{g} = \frac{h_1 + h_2 s + \ldots + h_n s^{n-1}}{\det A_s} \tag{3-15}$$

This result is needed in the following sufficiency proof.

Proof of Sufficiency

The functions $2\mathrm{Re}\underline{h}'A_\omega^{-1}\ \underline{g} - \rho\ \underline{g}'A_\omega^{*-1}\ RA_\omega^{-1}\ \underline{g}$ and $\underline{g}'A_\omega^{*-1}\ DA_\omega^{-1}\underline{g}$ are real rational functions of ω with numerators of degree less than their denominator, and therefore they go to zero as ω goes to infinity. They are continuous for finite ω, and hence they have finite upper and lower bounds. Let μ be the upper bound of $\underline{g}'A_\omega^{*-1}\ DA_\omega^{-1}\ \underline{g}$ and π be the lower bound of $2\mathrm{Re}\underline{h}'A_\omega^{-1}\ \underline{g} - \rho\ \underline{g}'A_\omega^{-1*}\ RA_\omega^{-1}\ \underline{g}$. Since $D > 0$, then $\mu > 0$. Hence

$$\begin{aligned} &\tau + 2\mathrm{Re}\underline{h}'A_\omega^{-1}\ \underline{g} - \rho\ \underline{g}'A_\omega^{-1*}\ RA_\omega^{-1}\ \underline{g} \\ &- \epsilon\ \underline{g}'A_\omega^{*-1}\ DA_\omega^{-1}\ \underline{g} \geq \tau + \pi - \epsilon\mu \end{aligned} \tag{3-16}$$

However, by (*) $\tau + \pi > 0$. Hence, if $\epsilon = 1/2(\tau + \pi)/\mu$, then

$$\begin{aligned} &\tau + 2\mathrm{Re}\underline{h}'A_\omega^{-1}\ \underline{g} - \rho\ \underline{g}'A_\omega^{-1*}\ RA_\omega^{-1}\ \underline{g} \\ &- \epsilon\underline{g}'A_\omega^{-1*}\ DA_\omega^{-1}\ g > 0 \end{aligned} \tag{3-17}$$

Let det $A_\omega = a(j\omega)$ which is a real polynomial of $j\omega$ with leading coefficient unity. The last three terms on the left side of (3-17) are of the form a polynomial in ω^2 divided by $|a(j\omega)|^2$. This is because they are either the real part of a function of $j\omega$ or the magnitude of such a function. Therefore, the left side of (3-17) can be written as

$$\tau + 2\mathrm{Re}\underline{h}'A_\omega^{-1}\underline{g} - \rho\,\underline{g}'A_\omega^{-1*}RA_\omega^{-1}\underline{g}$$

$$- \epsilon\,\underline{g}'A_\omega^{-1*}DA_\omega^{-1}\underline{g} = \frac{u(\omega^2)}{a(j\omega)a(-j\omega)} \tag{3-18}$$

where $u(\omega^2)$ is a polynomial of degree 2n with leading coefficient τ. However, by (3-17) $u(\omega^2)$ is always greater than zero for all ω. $u(\omega^2)$ is a real, positive, and even function of $j\omega$. By the spectrum factorization method of the Wiener theory of optimum linear systems (Lee 1960, p. 376), $u(\omega^2)$ can be written as

$$u(\omega^2) = \phi(-j\omega)\phi(j\omega) \tag{3-19}$$

where $\phi(j\omega)$ is a polynomial in $j\omega$ with real coefficients. Since the leading coefficient of $u(\omega^2)$ is τ, that of $\phi(j\omega)$ is $\sqrt{\tau}$, and the degree of $\phi(j\omega)$ is n. Therefore, $\phi(j\omega)/a(j\omega)$ can be written as $v(j\omega)/a(j\omega) + \sqrt{\tau}$ and (3-18) becomes

$$\tau + 2\mathrm{Re}\underline{h}'A_\omega^{-1}\underline{g} - \rho\underline{g}'A_\omega^{-1*}RA_\omega^{-1}\underline{g} - \epsilon\,\underline{g}'A_\omega^{-1*}DA_\omega^{-1}\underline{g}$$

$$= \left(\frac{v(j\omega)}{a(j\omega)} + \sqrt{\tau}\right)^* \left(\frac{v(j\omega)}{a(j\omega)} + \sqrt{\tau}\right) \tag{3-20}$$

$v(j\omega)$ is a polynomial of degree at most n-1. If $v_1, v_2, \ldots, v_n$ are the real coefficients of $v(j\omega)$; define $\underline{q}$ by

$$\underline{q}' = -(v_1 \; v_2 \; \ldots \; v_n) \tag{3-21}$$

Once $\underline{q}$ is known, the matrix P is obtained by (a) of the lemma. Since A is a stable matrix and $Q = \underline{q}\,\underline{q}' + \epsilon R + \epsilon D$ is positive definite, then use of the Liapunov theory for linear systems shows that if A is stable and Q is positive definite (or semidefinite), the matrix P which results from solving $A'P + PA = -Q$ must be positive definite.

This may seem to be a rather arbitrary definition for $\underline{q}$. However, this $\underline{q}$ is now shown to also satisfy (b) of the lemma by going back to the necessity part of the proof. First of all, as indicated previously, the matrix A and vector $\underline{g}$ can be assumed to have a certain form. Referring to (3-15), it is seen that

$$\frac{v(j\omega)}{a(j\omega)} = -\underline{q}'A_\omega^{-1}\,\underline{g} \tag{3-22}$$

Hence (3-20) becomes

$$\tau + 2\mathrm{Re}\underline{h}'A_\omega^{-1}\,\underline{g} - \epsilon\underline{g}'A_\omega^{-1*}\,RA_\omega^{-1}\,\underline{g} - \epsilon\,\underline{g}'A_\omega^{-1*}\,DA_\omega^{-1}\,\underline{g}$$

$$= (-\underline{q}'A_\omega^{-1}\,\underline{g} + \sqrt{\tau}\,)^*(-\underline{q}'A_\omega^{-1}\,\underline{g} + \sqrt{\tau}\,) \tag{3-23}$$

Multiplying out the right side gives

$$\tau + 2\mathrm{Re}\underline{h}'A_\omega^{-1}\,\underline{g} - \epsilon\underline{g}'A_\omega^{-1*}\,RA_\omega^{-1}\,\underline{g} - \epsilon\,\underline{g}'A_\omega^{-1*}\,DA_\omega^{-1}\,\underline{g}$$

$$= \underline{g}'A_\omega^{-1*}\,\underline{q}\,\underline{q}'A_\omega^{-1}\,\underline{g} - 2\mathrm{Re}\underline{q}'A_\omega^{-1}\,\underline{g} + \tau \tag{3-24}$$

The τ cancels; and going back to (3-12) in the necessity proof, solving it for $\underline{g}'A_\omega^{*-1}\ \underline{q}\ \underline{q}'A_\omega^{-1}\ \underline{g}$ and substituting this into (3-24) results in

$$2\mathrm{Re}\underline{h}'A_\omega^{-1}\ \underline{g} - \rho\ \underline{g}'A_\omega^{-1*}\ RA_\omega^{-1}\ \underline{g} - \epsilon\underline{g}'A_\omega^{-1*}\ DA_\omega^{-1}\ \underline{g}$$

$$= 2\mathrm{Re}\underline{g}'PA_\omega^{-1}\ \underline{g} - \rho\underline{g}'A_\omega^{*-1}\ RA_\omega^{-1}\ \underline{g} - \epsilon\ \underline{g}'A_\omega^{*-1}\ DA_\omega^{-1}\ \underline{g}$$

$$- \sqrt{\tau}\ 2\mathrm{Re}(\underline{q}'A_\omega^{-1}\ \underline{g})$$

Cancelling the proper terms and manipulating the remaining ones give

$$2\mathrm{Re}(P\underline{g} - \underline{h} - \sqrt{\tau}\ \underline{q})'A_\omega^{-1}\ \underline{g} = 0 \qquad (3\text{-}26)$$

For (3-28) to be true for all real ω, the vector in the parenthesis must be zero or

$$P\underline{g} - \underline{h} - \sqrt{\tau}\ \underline{q} = 0$$

This is just (b) of the lemma. Therefore, by starting with (*), a vector $\underline{q}$ was constructed and a matrix P found which satisfy (a) and (b). Therefore (*) is sufficient for the existence of the solution of (a) and (b).

In the work of Rekasius and Rowland, a result similar to Lemma 1 is used. It is actually the case where $R = \underline{r}\ \underline{r}'$. This can be stated as a corollary

Corollary 1: If the matrix $R = \underline{r}\ \underline{r}'$, then (*) is written

$$\tau + 2\mathrm{Re}\underline{h}'A_\omega^{-1}\ \underline{g} - \rho|\underline{r}'A_\omega^{-1}\ \underline{g}|^2 > 0 \qquad (3\text{-}29)$$

This corollary is used in Chapter 4.

Another special case is when β and ϵ are zero. The lemma then reduces to Kalman's lemma (1963) with the result that the less than sign is replaced by a less than or equal to sign, that is,

$$\tau + 2\mathrm{Re}\,\underline{h}'A_{\omega}^{-1}\,\underline{g} \geq 0 \tag{3-30}$$

This will also be of use in what follows.

Now that the lemma has been proven, it can be used to prove the Popov criterion by relating the Popov criterion to the Second Method of Liapunov. The simplest particular case is treated first, and a theorem relating the Popov criterion to the Second Method is stated. This particular statement of the theorem follows Lefschetz (1965) and is used because it has the condition that $-\dot{V}$ be positive definite in it. Other statements of this type of theorem (Kalman 1963) have the condition that $-\dot{V}$ be only positive semidefinite. The definite $\dot{V}$ is preferred here since applications are to be made to time-varying systems, where $-\dot{V}$ must be positive definite to conclude asymptotic stability.

Theorem 3-2: A necessary and sufficient condition in order that, with V_1 as above, both V_1 and $-\dot{V}_1$ are positive definite for all $\underline{x}$, σ and admissible $f(\sigma)$ is that the Popov inequality

$$\mathrm{Re}(2\alpha\gamma + j\omega\beta)G(j\omega) + \frac{2\alpha\gamma}{k} > 0 \tag{3-31}$$

holds for all real ω together with $\tau > 0$ where

$$\frac{1}{\tau} = \beta\underline{C}'\underline{b} + \beta\gamma + \frac{2\alpha\gamma}{k} .$$

When these properties are satisfied the system (2-5) is absolutely stable.

The Popov inequality above is obtained from the original Popov condition (3-1) by letting $2\alpha\gamma = 1$.

The proof of the theorem requires putting $-\dot{V}_1$ in a form such that the lemma can be applied. For convenience the system equations are repeated. They are

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ u &= -f(\sigma) \\ \dot{\xi} &= f(\sigma) \\ \sigma &= \underline{c}'\underline{x} - \gamma\xi \end{aligned} \tag{2-5}$$

The Liapunov function is

$$V_1 = \underline{x}'P\underline{x} + \alpha(\sigma - c'\underline{x})^2 + \beta \int_0^\sigma f(z)dz \tag{3-7}$$

but $(\sigma - \underline{c}'\underline{x})^2 = \gamma^2\xi^2$ so that

$$\dot{V}_1 = \underline{x}'(A'P + PA)\underline{x} + 2\underline{x}'P\underline{b}u + 2\gamma^2\alpha\xi\dot{\xi} + \beta f(\sigma)(\underline{c}'\dot{\underline{x}} - \gamma\dot{\xi})$$

Substituting $f(\sigma)$ for $\dot{\xi}$, $\underline{c}'\underline{x} - \sigma$ for $\gamma\xi$ and $-f(\sigma)$ for u, and collecting terms gives

$$\dot{V}_1 = \underline{x}'(A'P + PA)\underline{x} - \underline{x}'f(\sigma)(2P\underline{b} - 2\gamma\alpha\underline{c} - \beta A'\underline{c})$$

$$- (\beta\underline{c}'\underline{b} + \beta\gamma)f(\sigma)^2 - 2\gamma\alpha f(\sigma)\sigma$$

The quantity $\lambda(\sigma) = 2\gamma\alpha(\sigma - \frac{f(\sigma)}{k})f(\sigma)$ is now added and subtracted from $\dot{V}_1$ giving

$$\dot{V}_1 = \underline{x}'(A'P + PA)\underline{x} - \underline{x}'f(\sigma)(2P\underline{b} - 2\gamma\alpha\underline{c} - \beta A'\underline{c})$$

$$- (\beta\underline{c}'\underline{b} + \beta\gamma + \frac{2\gamma\alpha}{k})f(\sigma)^2 - \lambda(\sigma) \tag{3-32}$$

The quantity $\lambda(\sigma)$ is always positive since γ and α must be positive, and $(\sigma - \frac{f(\sigma)}{k})f(\sigma)$ is also positive. This is because of the inequality $0 < f(\sigma)/\sigma \leq k$. Multiplying the inequality by the positive quantity $\frac{\sigma f(\sigma)}{k}$ produces $0 < \frac{f(\sigma)^2}{k} \leq \sigma f(\sigma)$ so that $\sigma f(\sigma) - f(\sigma)^2/k = (\sigma - f(\sigma)/k) f(\sigma) \geq 0$. Letting $A'P + PA = -Q$ and writing $-\dot{V}_1$ gives

$$-\dot{V}_1 = \underline{x}'Q\underline{x} + \underline{x}'f(\sigma)(2P\underline{b} - 2\gamma\alpha\underline{c} - \beta A'\underline{c})$$

$$+ (\beta\underline{c}'\underline{b} + \beta\gamma + \frac{2\gamma\alpha}{k})f(\sigma)^2 + \lambda(\sigma) \tag{3-33}$$

In order to apply the lemma, $-\dot{V}_1$ should be forced to assume a form such that the solution of a set of algebraic equations shows $-\dot{V}_1$ to be positive definite. The proper form is

$$-\dot{V}_1 = (\underline{q}'\underline{x} + f(\sigma)/\sqrt{\tau})^2 + \underline{x}'\epsilon D\underline{x} + \lambda(\sigma) \tag{3-34}$$

where D is positive definite. If $-\dot{V}_1$ can be made to have this form, it is positive definite, and since V is positive definite, the system is absolutely stable. Multiplying (3-34) out gives

$$-\dot{V}_1 = \underline{x}'\underline{q}\,\underline{q}'\underline{x} + 2f(\sigma)\underline{x}'\underline{q}/\sqrt{\tau} + \frac{f(\sigma)^2}{\tau} + \underline{x}'\epsilon D\underline{x} + \lambda(\sigma) \tag{3-35}$$

Equating the proper coefficients in (3-35) and (3-33) leads to the following set of algebraic equations.

$$Q = \underline{q}\,\underline{q}' + \epsilon D = -(A'P + PA) \tag{3-36}$$

$$2\underline{q}/\sqrt{\tau} = 2P\underline{b} - 2\gamma\alpha\underline{c} - \beta A'\underline{c} \tag{3-37}$$

$$\frac{1}{\tau} = \beta\underline{c}'\underline{b} + \beta\gamma + \frac{2\gamma\alpha}{k} \tag{3-38}$$

The lemma can now be applied to this set of equations, and it gives necessary and sufficient conditions for $\underline{q}$ and $P > 0$ to exist. In other words the lemma gives necessary and sufficient conditions for the existence of V_1 and $-\dot{V}_1$ to exist and be positive definite.

In order to use the lemma the expression for $\sqrt{\tau}\ \underline{q}$ is needed.

$$\sqrt{\tau}\ \underline{q} = P(\tau\underline{b}) - \frac{1}{2}\tau(2\gamma\alpha\underline{c} + \beta A'\underline{c}) = P\underline{g} - \underline{h} \tag{3-39}$$

Now that $\underline{g}$, $\underline{h}$ and τ have been identified, the condition for the existence of a solution to the set of equations is given. This condition is now also a condition for the existence of a Liapunov function so that it is a stability criterion. The condition is

$$\tau + 2\mathrm{Re}\underline{h}'A_\omega^{-1}\underline{g} > 0 \tag{3-40}$$

This is (*) with the term $\rho R = 0$. Substituting the proper quantities for $\underline{h}'$ and $\underline{g}$ into (3-40) gives

$$\tau + 2\mathrm{Re}\frac{1}{2}\tau(2\gamma\alpha\underline{c} + \beta A'\underline{c})'A_\omega^{-1}\tau\underline{b} > 0$$

and

$$\frac{1}{\tau} + 2\mathrm{Re}(\alpha\gamma\underline{c}'A_\omega^{-1}\underline{b} + \frac{1}{2}\beta\underline{c}'AA_\omega^{-1}\underline{b}) > 0 \tag{3-41}$$

Now A_ω was previously defined as $A_\omega = j\omega I - A$. Therefore, $A = j\omega I - A_\omega$ and A_ω^{-1} exists for all real ω since the eigenvalues of A are all in the left half plane. Therefore, postmultiplying A by A_ω^{-1} gives

$$AA_\omega^{-1} = j\omega A_\omega^{-1} - I \tag{3-42}$$

Substituting this into (3-41) along with the expression for $1/\tau$, i.e., (3-38) gives

$$\beta \underline{c}'\underline{b} + \beta\gamma + \frac{2\gamma\alpha}{k} + 2\mathrm{Re}(\alpha\ \gamma\ \underline{c}'A_{\omega}^{-1}\ \underline{b}$$

$$+ \frac{1}{2}\ \beta\underline{c}'\ j\omega A_{\omega}^{-1}\ \underline{b} - \frac{1}{2}\ \beta\underline{c}'\underline{b}) > 0$$

or

$$\beta\gamma + \frac{2\gamma\alpha}{k} + \mathrm{Re}(2\gamma\alpha + j\omega\beta)\underline{c}'A_{\omega}^{-1}\ \underline{b} > 0 \qquad (3\text{-}43)$$

This is the exact expression that appears in Lefschetz (1965, p. 125). If use if made of the fact that $\mathrm{Re}(2\alpha\gamma + j\omega\beta)\frac{\gamma}{j\omega} = \beta\gamma$, then (3-43) can be rewritten as

$$\frac{2\gamma\alpha}{k} + \mathrm{Re}(2\gamma\alpha + j\omega\beta)(\underline{c}'A_{\omega}^{-1}\ \underline{b}\ +\ \frac{\gamma}{j\omega}) > 0 \qquad (3\text{-}44)$$

But, for the simplest particular case, the term $\underline{c}'A_{\omega}^{-1}\ \underline{b} + \gamma/j\omega$ is just the transfer function of the linear part of the system so that (3-44) becomes

$$\frac{2\gamma\alpha}{k} + \mathrm{Re}(2\gamma\alpha + j\omega\beta)G(j\omega) > 0 \qquad (3\text{-}45)$$

which is the inequality which appears in Theorem 3-2. If now the term $2\gamma\alpha$ is put equal to one, the result is just the basic Popov inequality (3-1), and β can be found by using the geometrical approach.

Next the relationship between the Second Method and the Popov criterion is given for the principal case of the system (2.3). The system equations are

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ u &= -f(\sigma) \qquad\qquad (2\text{-}3) \\ \sigma &= \underline{c}'\underline{x} \end{aligned}$$

and the Liapunov function is

$$V_0 = \underline{x}'P\underline{x} + \beta \int_0^{\sigma} f(z)dz \tag{3-6}$$

Taking $\dot{V}_0$ gives

$$\dot{V}_0 = \dot{\underline{x}}'P\underline{x} + \underline{x}'P\dot{\underline{x}} + \beta f(\sigma)\dot{\sigma} = 2\underline{x}'P\dot{\underline{x}} + \beta f(\sigma)c'\dot{\underline{x}} \tag{3-46}$$

Factoring out the $\dot{\underline{x}}$ gives

$$-\dot{V}_0 = -(2P\underline{x} + \beta\underline{c}f(\sigma))'(A\underline{x} - \underline{b}f(\sigma)) \tag{3-47}$$

Some people tried to get a positive definite quadratic form in $\underline{x}$ and $f(\sigma)$ directly from (3-46)(Aizerman and Gantmacher 1964, p. 20). Since $A\underline{x} - \underline{b}f$ can be zero for $\underline{x}$ and $f(\sigma)$ not zero, $-\dot{V}_0$ can at best be semi-definite if treated as a quadratic form in $\underline{x}$ and f. This problem did not occur in the simplest particular case because of the quantities ξ and $\dot{\xi}$ also occurring in the $-\dot{V}_1$ equation. However, the difficulty is easily avoided by adding and subtracting the term $\lambda(\sigma) = \delta(\sigma - \frac{f(\sigma)}{k})f(\sigma)$. The result is

$$-\dot{V}_0 = S(x,f(\sigma)) + \varepsilon(\sigma - \frac{f(\sigma)}{k})f(\sigma) \tag{3-48}$$

where the function $S(x,f(\sigma))$ must be positive definite if $\dot{V}_0$ is to be negative definite.

> Theorem 3-3: Necessary and sufficient conditions for V_0 to be positive definite as a function of $\underline{x}$ for all admissible functions $f(\sigma)$ and S positive definite as a quadratic form in x and $f(\sigma)$ is the Popov inequality

$$\mathrm{Re}(\delta + j\omega\beta)G(j\omega) + \frac{\varepsilon}{k} > 0 \tag{3-49}$$

for some $\beta \geq 0$, some positive δ, and all real ω. When these conditions are fulfilled both V_0 and $-\dot{V}_0$ are positive definite and the system is absolutely stable.

Aizerman and Gantmacher (1964) point out that (3-49) is necessary and sufficient for the existence of a Liapunov function constructed by the above S-procedure, but that there exist other Liapunov functions of the form V_0 which cannot be determined by the S-procedure. However, Yakubovich (1964b) has shown that there does not exist an absolutely stable system of the form (2-3) for which the fact of absolute stability can be established by using a Lurie type Liapunov function, but cannot be established by using the S-procedure.

The proof of the theorem proceeds in an entirely similar manner to the previous case. Putting the system equations into (3-46) gives

$$\dot{V}_0 = \underline{x}'(A'P + PA)\underline{x} - 2\underline{x}'P\underline{b}f(\sigma) + \beta f(\sigma)\underline{c}'A\underline{x} - \beta\underline{c}'\underline{b}f(\sigma)^2 \qquad (3\text{-}50)$$

Letting $A'P + PA = -Q$, adding and subtracting $\lambda(\sigma)$, and writing $-\dot{V}_0$ give

$$-\dot{V}_0 = \underline{x}'Q\underline{x} + (2b'P - \beta\underline{c}'A - \delta\underline{c}')f(\sigma)\underline{x} + (\frac{\delta}{k} + \beta\underline{c}'\underline{b})\ f(\sigma)^2 + \lambda(\sigma) \qquad (3\text{-}51)$$

Again $-\dot{V}_0$ should have the same form as given in the previous derivation, i.e., (3-34). Equating the like terms in (3-35) and (3-51) results in the equations

$$Q = \epsilon D + \underline{q}\ \underline{q}' = -(A'P + PA) \qquad (3\text{-}52)$$

$$2\underline{q}/\sqrt{\tau} = 2P\underline{b} - \beta A'\underline{c} - \beta A'\underline{c} - \delta\underline{c} \tag{3-53}$$

$$\frac{1}{\tau} = \frac{\delta}{k} + \beta\underline{c}'\underline{b} \tag{3-54}$$

so that for $D > 0$, the solution of this set of equations guarantees absolute stability as before. The lemma requires

$$\tau + 2\mathrm{Re}\,\underline{h}'A_{\omega}^{-1}\,\underline{g} > 0 \tag{3-40}$$

where now $\underline{h} = \frac{1}{2}\tau(\beta A'\underline{c} + \delta\underline{c})$ and $\underline{g} = \tau\underline{b}$ so that

$$\tau + 2\mathrm{Re}(\tfrac{1}{2}\tau(\beta A'\underline{c} + \delta\underline{c})'A_{\omega}^{-1}\,\tau\,\underline{b}) > 0 \tag{3-55}$$

or

$$\frac{1}{\tau} + \mathrm{Re}(\beta\underline{c}'AA_{\omega}^{-1}\,\underline{b} + \delta\underline{c}'A_{\omega}^{-1}\,\underline{b}) > 0 \tag{3-56}$$

Making the substitutions for $1/\tau$ and AA_{ω}^{-1} gives

$$\frac{\delta}{k} + \beta\underline{c}'\underline{b} + \mathrm{Re}(\beta\underline{c}'j\omega A_{\omega}^{-1}\,\underline{b} - \beta\underline{c}'\underline{b} + \delta\underline{c}'A_{\omega}^{-1}\,\underline{b}) > 0$$

or

$$\frac{\delta}{k} + \mathrm{Re}(j\omega\beta + \delta)\underline{c}'A_{\omega}^{-1}\,\underline{b} > 0 \tag{3-57}$$

which again is Lefschetz's result. Since $\underline{c}'A_{\omega}^{-1}\,\underline{b} = G(j\omega)$, this can be written as

$$\frac{\delta}{k} + \mathrm{Re}(j\omega\beta + \delta)\,G\,(j\omega) > 0 \tag{3-58}$$

which is the inequality in the theorem. Letting $\delta = 1$ again gives the basic Popov inequality (3-1).

The basic Popov criterion and its relationship with the Second Method of Liapunov has now been given. In the next chapter the use of these results for time-varying systems is discussed.

Chapter 4

FREQUENCY CRITERIA FOR TIME-VARYING SYSTEMS

4.1 Introduction

In this chapter Lemma 1 is used to derive various frequency domain stability criteria for a class of nonlinear, time-varying systems. The second section includes the work of Rozenvasser (1963), who treated the principal case, and this is extended to the simplest particular case. The Liapunov function used in this section is a quadratic form of the state variables.

The third section examines the work of Bongiorno (1963), Sandberg (1964), and Narendra and Goldwyn (1964) on the subject of time-varying systems, and shows how their work is related to the Popov criterion. The geometric interpretation of these various results is given in the fourth section. The fifth section consists of two examples illustrating the previous results. The frequency domain criteria are applied to the equation which arises from a RLC circuit with time-varying capacitance, and also to the nuclear reactor kinetics equations.

The last section considers how the integral term can be put back into the Liapunov function. This results in extensions of the previous results of the chapter. The RLC circuit example is reworked to show when these new results are applicable.

4.2 Stability of Time-Varying Systems Using the Popov Criterion

The Popov criterion was shown to be valid for the principal case of the time-varying system by Rozenvasser (1963). The development is similar to that of the previous chapter, except that in this case the quantity β is zero. Thus, the Liapunov function connected with this development is just a quadratic form of the state variables. The reason for eliminating the integral term is that the integral term is time-varying, and only creates additional complications when the time derivative is taken.

The system equations are

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ u &= -f(\sigma,t), \quad 0 \leq f(\sigma,t)/\sigma \leq k \\ \sigma &= \underline{c}'\underline{x} \end{aligned} \tag{2-3}$$

and the Liapunov function is

$$V = \underline{x}'P\underline{x} \tag{4-1}$$

Taking the time derivative and adding and subtracting $\lambda(\sigma,t) = f(\sigma,t)(\sigma - f(\sigma,t)/k) \geq 0$ to it gives

$$\dot{V} = \underline{x}'(A'P + PA)\underline{x} - 2\underline{x}'P\underline{b}f(\sigma,t) + f(\sigma,t)(\sigma - f(\sigma,t)/k) - \lambda(\sigma,t) \tag{4-2}$$

Letting $A'P + PA = -Q$, $\sigma = \underline{c}'\underline{x}$ and writing $-\dot{V}$ gives

$$-\dot{V} = \underline{x}'Q\underline{x} + \underline{x}'f(\sigma,t)(2P\underline{b} - \underline{c}) + f(\sigma,t)^2/k + \lambda(\sigma,t) \tag{4-3}$$

Once again $-\dot{V}$ should take the form of (3-34) to insure that it is positive definite. Equating the like terms in (3-35) and (4-3) gives the following equations

$$Q = \underline{q}\,\underline{q}' + \epsilon D = -(A'P + PA) \tag{4-4}$$

$$2\underline{q}/\sqrt{\tau} = 2P\underline{b} - \underline{c} \tag{4-5}$$

$$\tau = k \tag{4-6}$$

Applying Lemma 1 with $\underline{g} = k\underline{b}$ and $\underline{h} = k\underline{c}/2$ gives

$$k + 2\mathrm{Re}\,\frac{1}{2}\,k\underline{c}'A_\omega^{-1}\,k\underline{b} > 0$$

or

$$\frac{1}{k} + \mathrm{Re}\,\underline{c}'A_\omega^{-1}\,\underline{b} > 0 \tag{4-7}$$

But $\underline{c}'A_\omega^{-1}\,\underline{b} = G(j\omega)$ so that (4-7) can be written as

$$\frac{1}{k} + \mathrm{Re}G(j\omega) > 0 \tag{4-8}$$

(4-8) is just the Popov criterion for $\beta = 0$ as given in (3-1), and has the same geometric interpretation as given in Fig. 4.

The derivation is also repeated for the simplest particular case, since this was not considered by Rozenvasser. The system equations are

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ u &= -f(\sigma,t) \\ \dot{\xi} &= f(\sigma,t) \qquad 0 < f(\sigma,t)/\sigma \le k \\ \sigma &= \underline{c}'\underline{x} - \gamma\,\xi \end{aligned} \tag{4-9}$$

and the Liapunov function is

$$V = \underline{x}'P\underline{x} + \alpha(\sigma - \underline{c}'\underline{x})^2 \tag{4-10}$$

Taking $\dot{V}$, and adding and subtracting $\lambda(\sigma,t)$ leads to the equation

$$-\dot{V} = \underline{x}'Q\underline{x} + \underline{x}'f(\sigma,t)(2P\underline{b} - 2\alpha\,\gamma\,\underline{c})$$

$$+ \frac{2\alpha\gamma}{k}\, f(\sigma,t)^2 + \lambda(\sigma,t) \tag{4-11}$$

Comparing (4-11) with (3-35) and applying the usual lemma with $\underline{g} = k\underline{b}$, $\underline{h} = k\,\alpha\,\gamma\,\underline{c}$, and $1/\tau = 2\alpha\gamma/k$ gives

$$\frac{2\alpha\gamma}{k} + 2\text{Re}\ \alpha\,\gamma\ \underline{c}'A_\omega^{-1}\ \underline{b} > 0 \tag{4-12}$$

But $\text{Re}\frac{\gamma}{j\omega}(2\alpha\gamma) = 0$. Adding this to (4-12) gives

$$\frac{2\alpha\gamma}{k} + \text{Re}2\alpha\gamma(\underline{c}'A_\omega^{-1}\ \underline{b} + \frac{\gamma}{j\omega}) > 0 \tag{4-13}$$

or for $2\alpha\gamma = 1$

$$\frac{1}{k} + \text{Re}G(j\omega) > 0 \tag{4-14}$$

Hence, the Popov criterion of $\beta = 0$ holds for the case of a nonlinear time-varying element in the simplest particular case also.

4.3 Other Work

There has been work by other investigators which gives essentially the same results. Bongiorno (1963) derives his results for linear systems with a periodic variation of the time-varying element, using a combination of Floquet theory and Fourier analysis. Sandberg (1964) derives his results using functional analysis. Narendra and Goldwyn (1964) use the Second Method of Liapunov to get similar results. This section shows that these results can be derived using Lemma 1, and they are essentially the same as the Popov criterion.

In the work which follows, only the principal case is considered, and the nonlinearity $f(\sigma,t)$ is assumed to be in some sector $[k_1, k_2]$, and it is written as $f(\sigma,t) = k(\sigma,t)\sigma$. Putting the system equations (2-3) into the time derivative of the Liapunov function (4-1) gives

$$\dot{V} = \underline{x}'(A'P + PA)\underline{x} + 2\underline{x}'P\underline{b}u \tag{4-15}$$

Letting $A'P + PA = -Q$ and replacing u by $u = -k(\sigma,t)\sigma$ gives

$$\dot{V} = -\underline{x}'(Q + 2P\underline{b}\ \underline{c}'k(\sigma,t))\underline{x} \tag{4-16}$$

Analogous to the previous work, $-\dot{V}$ is forced to be positive definite.

$$-\dot{V} = \underline{x}'(\epsilon\ D + (\underline{q} + k(\sigma,t)\underline{c})\ (\underline{q} + k(\sigma,t)\underline{c})'$$

$$+ (k_2 - k(\sigma,t))(k(\sigma,t) - k_1)\underline{c}\ \underline{c}')\underline{x} \tag{4-17}$$

This is the $\dot{V}$ used by Narendra and Goldwyn (1964). Multiplying this expression out gives

$$-\dot{V} = \underline{x}'(\epsilon\ D + \underline{q}\ \underline{q}' - k_1k_2\underline{c}\ \underline{c}' + 2\underline{q}\ \underline{c}'k(\sigma,t)$$

$$+ (k_1 + k_2)\underline{c}\ \underline{c}'k(\sigma,t))\underline{x} \tag{4-18}$$

Comparing (4-16) and (4-18) gives the following set of equations

$$Q = \epsilon\ D + \underline{q}\ \underline{q}' - k_1k_2\underline{c}\ \underline{c}' \tag{4-19}$$

$$P\underline{b} = \underline{q} + \frac{1}{2}(k_1 + k_2)\ \underline{c} \tag{4-20}$$

The corollary to Lemma 1 can be applied to this set of equations getting

$$1 + 2\mathrm{Re}\frac{1}{2}(k_1 + k_2)\underline{c}'A_{\omega}^{-1}\,\underline{b} + k_1k_2|\underline{c}'A_{\omega}^{-1}\,\underline{b}|^2 > 0 \qquad (4\text{-}21)$$

or

$$1 + (k_1 + k_2)\mathrm{Re}G(j\omega) + k_1k_2|G(j\omega)|^2 > 0 \qquad (4\text{-}22)$$

This is the result achieved by Narendra and Goldwyn. It should be pointed out that (4-21) is true only when (4-19) is positive definite.

The results of Bongiorno can be obtained by setting up the system equations such that $k_1 = -k_2$. In that case (4-22) becomes

$$1 - k_2^2|G(j\omega)|^2 > 0$$

or

$$k_2|G(j\omega)| < 1 \qquad (4\text{-}23)$$

Bongiorno attained this result by means of a completely different derivation.

The results of Sandberg can also be obtained from (4-22). First divide (4-22) by $|G(j\omega)|^2 = G(j\omega)G(-j\omega)$ and then add and subtract $\frac{1}{4}(k_1 + k_2)^2$. The result is

$$\frac{1}{G(j\omega)G(-j\omega)} + \frac{k_1 + k_2}{2}\left(\frac{1}{G(j\omega)} + \frac{1}{G(-j\omega)}\right)$$

$$+ \frac{(k_1 + k_2)^2}{4} - \frac{(k_1 + k_2)^2}{4} + k_1k_2 > 0 \qquad (4\text{-}24)$$

This can be rewritten as

$$\left(\frac{1}{G(j\omega)} + \frac{k_1 + k_2}{2}\right)\left(\frac{1}{G(-j\omega)} + \frac{k_1 + k_2}{2}\right) - \frac{(k_2 - k_1)^2}{4} > 0$$

or

$$\left|\frac{1}{G(j\omega)} + \frac{k_1 + k_2}{2}\right|^2 > \frac{(k_2 - k_1)^2}{4} \tag{4-25}$$

The final result being

$$\frac{k_2 - k_1}{2}\left|\frac{G(j\omega)}{1 + (\frac{k_1 + k_2}{2})G(j\omega)}\right| < 1 \tag{4-26}$$

which is Sandberg's criterion.

It is also seen that, by letting $k_1 = 0$ and $k_2 = k$ in (4-22), the result is

$$\mathrm{Re}G(j\omega) + \frac{1}{k} > 0 \tag{4-8}$$

which is just the Popov criterion derived in section 4.2.

Actually (4-22) can be derived in another manner, that is by just using the standard Popov equation and rotating the nonlinear sector. Starting with the original set of equations

$$\begin{aligned} &\dot{\underline{x}} = A\underline{x} + \underline{b}u \\ &u = -f(\sigma,t), \quad 0 \le f(\sigma,t)/\sigma \le k \\ &\sigma = \underline{c}'\underline{x} \\ &G(s) = \underline{c}'(sI - A)^{-1}\,\underline{b} \end{aligned} \tag{2-3}$$

The nonlinear sector is rotated by the transformation $f = f_1 - k_1\sigma$. This means that $k_1 \le f_1/\sigma \le k + k_1 = k_2$ or $k = k_2 - k_1$. Substituting this into (2-3) gives

$$
\begin{aligned}
&\underline{\dot{x}} = A_1\underline{x} + \underline{b}u_1 \\
&u_1 = -f_1(\sigma,t), \quad k_1 \leq f_1(\sigma,t)/\sigma \leq k_2 \\
&\sigma = \underline{c}'\underline{x} \\
&G_1(s) = \underline{c}'(sI - A_1)^{-1}\,\underline{b}
\end{aligned}
\tag{4-27}
$$

where $A_1 = (A + k_1\underline{b}\,\underline{c}')$ and $u_1 = u - k_1\sigma$. To find the relation between $G(s)$ and $G_1(s)$, the equation relating A and A_1 is used in the original system equations. Taking the Laplace transform gives

$$(sI - A)\underline{x}(s) = \underline{b}u(s)$$

$$(sI - A_1 + k_1\,\underline{b}\,\underline{c}')\underline{x}(s) = \underline{b}u(s)$$

$$\underline{x}(s) + (sI - A_1)^{-1}\,k_1\,\underline{b}\,\underline{c}'x(s) = (sI - A_1)^{-1}\,\underline{b}u(s)$$

$$\underline{c}'\underline{x}(s) + \underline{c}'(sI - A_1)^{-1}\,\underline{b}k_1\underline{c}'\underline{x}(s) = \underline{c}'(sI - A_1)^{-1}\,\underline{b}u(s)$$

or

$$\sigma(s) + G_1(s)k_1\sigma(s) = G_1(s)u(s)$$

$$G(s) = \sigma(s)/u(s) = \frac{G_1(s)}{1 + k_1G_1(s)} \tag{4-28}$$

Putting this into the Popov expression (3-11) along with $k = k_2 - k_1$ gives

$$\frac{1}{k_2 - k_1} + \mathrm{Re}\,\frac{G_1(j\omega)}{1 + k_1G_1(j\omega)} > 0$$

Rewriting this inequality gives

$$\frac{1}{k_2 - k_1} + \frac{1}{2}\,\frac{G_1(j\omega)}{1 + k_1G_1(j\omega)} + \frac{1}{2}\,\frac{G_1(j\omega)}{1 + k_1G_1(-j\omega)} > 0$$

Multiplying through by $(k_2 - k_1)(1 + k_1G_1(j\omega))(1 + k_1G_1(-j\omega))$ does not change the inequality, and, by suitable grouping of terms, gives (4-22). The next section gives the geometric meaning of these various criteria.

4.4 Geometric Interpretations

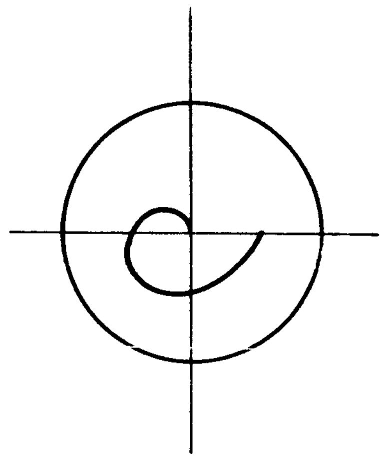
The following geometric interpretations can be put on (4-23), (4-26) and (4-8). For (4-23) it is obvious that the system is stable if the frequency locus is always inside the unit circle (Fig. 5a). (4-8) is just the usual geometric interpretation of the Popov criterion with $\beta = 0$ (Fig. 5d). (4-26) requires some work to interpret.

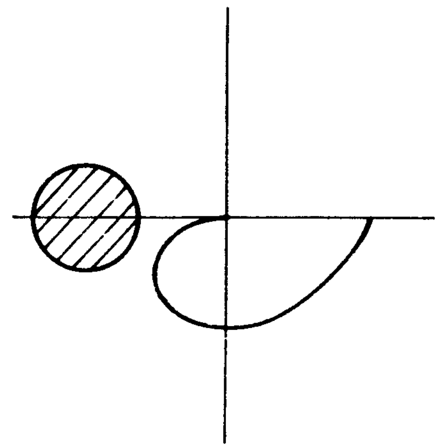
For (4-26) there are two cases to consider, $k_1 > 0$ and $k_1 < 0$. When $k_1 > 0$, the system is stable if the locus of $G(j\omega)$ is always outside the circle of radius $(k_2 - k_1)/2k_1k_2$ centered at $(-(k_1 + k_2))/2k_1k_2,0)$ (Fig. 5b), and for $k_1 < 0$ the system is stable if the locus of $G(j\omega)$ is always inside the circle of radius $(k_2 - k_1)/2k_1k_2$ centered at $(-(k_1 + k_2)/2k_1k_2,0)$ (Fig. 5c). This result is obtained by first squaring (4-26) and cross multiplying, getting

$$\left| 1 + \frac{k_1 + k_2}{2} G(j\omega) \right|^2 > \frac{(k_1 - k_2)^2}{4} \left| G(j\omega) \right|^2$$

Letting $G(j\omega) = x + jy$ results in, after sufficient manipulation, the inequality

$$1 + (k_1 + k_2)x + k_1k_2x^2 + k_1k_2y^2 > 0$$

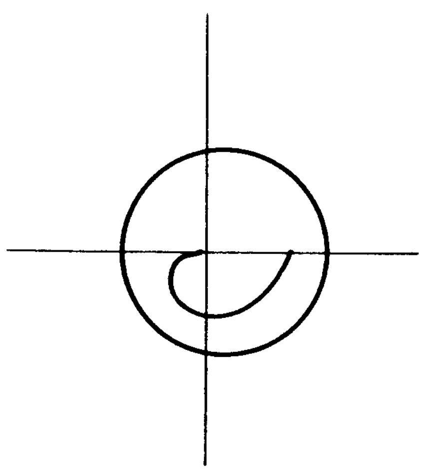


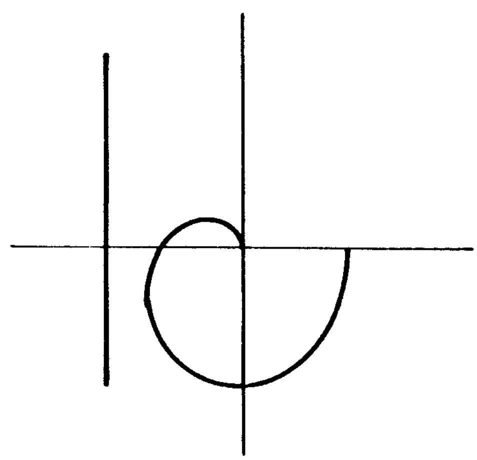


a) $-k \leq f(\sigma,t)/\sigma \leq k$

b) $k_1 \leq f(\sigma,t)/\sigma \leq k_2$

$k_1 > 0$





c) $k_1 \leq f(\sigma,t)/\sigma \leq k_2$

$k_1 < 0$

d) $0 \leq f(\sigma,t)/\sigma \leq k$

Fig. 5. Stability Criteria for Time-varying Systems.

Examples of Stable Systems.

When this is divided by k_1k_2, there are two results

$$\frac{1}{k_1k_2} + \frac{k_1 + k_2}{k_1k_2}x + x^2 + y^2 \quad \begin{matrix} > 0, k_1 > 0 \\ < 0, k_1 < 0 \end{matrix}$$

If the square is completed in x, the result is

$$\left(x + \frac{k_1 + k_2}{2k_1k_2}\right)^2 + y^2 \quad \begin{matrix} > \\ < \end{matrix} \quad \frac{(k_2 - k_1)^2}{4(k_1k_2)^2} \tag{4-29}$$

If the inequality signs are replaced by equality signs, (4-29) is the equation of a circle with the stability information obtained as indicated above.

All four geometric interpretations are illustrated in Fig. 5. It may be that one of these versions of the stability criterion is easier to apply than the others for a specific problem. This is illustrated in the examples which follow.

4.5 Examples and Discussion

In this section two examples are worked which illustrate the above stability criteria. The first example is the Mathieu equation, which arises from a series RLC circuit with a periodically varying capacitor. One reason for using this equation is that it has been studied extensively (McLachlan 1953), and its exact stability properties are known. This enables a comparison to be made with the sufficient results which are obtained here. The second example is concerned with nuclear reactor stability.

Example 4-1

The differential equation for this example is

$$\ddot{x} + 2\,\zeta\,\dot{x} + (1 - 2q\cos 2t)x = 0$$

This equation can be put into matrix form in two ways:

Case 1

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -0.1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} 2q(\cos 2t)x_1$$

Case 2

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -(1-2q) & -2\zeta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix} 2q(1-\cos 2t)x_1$$

The transfer function for Case 1 is $G_1(s) = 1/(s^2 + 2\zeta s + 1)$ with $-2q \leq f_1(x_1,t)/x_1 \leq 2q$, while for Case 2, $G_2(s) = 1/(s^2 + 2\zeta s + 1 - 2q)$ with $0 \leq f_2(x_1,t)/x_1 \leq 4q$. For Case 1 the Bongiorno criterion (4-23) is used and for Case 2 the Popov criterion (4-8) is used.

Case 1

$$K|G(j\omega)| < 1 \tag{4-23}$$

$$|G(j\omega)|^2 = \frac{1}{(1-\omega^2)^2 + 4\zeta^2\omega^2}$$

Setting the derivative with respect to ω of the denominator of $|G(j\omega)|^2$ equal to zero, to find out the frequency at which it is minimum, gives

$$2(1 - \omega^2)2\omega + 8\zeta^2 \omega = 0$$

$$\omega = 0 \qquad \zeta > .707$$

$$\omega = \sqrt{1 - \zeta^2} \qquad \zeta < .707$$

Looking only at the low damping case and substituting

$\omega = \sqrt{1 - \zeta^2}$ into $|G(j\omega)|^2$ gives

$$|G(j\omega)|^2 = \frac{1}{(2\zeta)^2 + 4\zeta^2(1 - 2\zeta^2)} = \frac{1}{4\zeta^2(1 - \zeta^2)}$$

Therefore $|G(j\omega)|$ max $= 1/2\zeta\sqrt{1 - \zeta^2}$. Putting this into (4-23) with $k = 2q$ gives

$$\frac{2q}{2\zeta\sqrt{1 - \zeta^2}} < 1$$

$$q < \zeta\sqrt{1 - \zeta^2}$$

Therefore, if q is less than $\zeta\sqrt{1 - \zeta^2}$, the system is known to be stable.

Case 2

$$\frac{1}{k} + \mathrm{Re}G(j\omega) > 0 \tag{4-8}$$

$$\frac{1}{4q} + \mathrm{Re}\,\frac{1}{1 - 2q - \omega^2 + 2\omega j\zeta} > 0$$

Calculating the real part gives

$$\frac{1}{4q} + \mathrm{Re}\,\frac{1 - 2q - \omega^2}{(1 + 2q - \omega^2)^2 + (2\zeta\omega)^2} > 0$$

Multiplying by $4q((1 - 2q - \omega^2)^2 + (2\zeta\omega)^2)$ does not change the inequality, and results in the expression

$$(-\omega^2 + 1)^2 - 4q^2 + 4\zeta^2\omega^2 > 0$$

Finding the minimum of value of this with respect to ω again gives $\omega = \sqrt{1 - 2\zeta^2}$ for $\zeta < .707$. Putting this into the previous expression gives

$$(-1 + 2\zeta^2 + 1)^2 - 4q^2 + 4\zeta^2(1 - 2\zeta^2) > 0$$

$$q^2 < \zeta^2(1 - \zeta^2)$$

$$q < \zeta\sqrt{1 - \zeta^2}$$

which is the same result as Case 1, as it should be.

For this simple example there does not seem to be any noticeable advantage of one stability criterion over the other. However, for higher order equations of the type

$$\overset{(n)}{x} + a_n \overset{(n-1)}{x} + \ldots + a_2\dot{x} + a_1(t)x = 0$$

the stability criterion given by (4-23) is easier to apply when finding how large the amplitude variation in $a_1(t)$ can be while being sure the system is stable. This is seen by observing the differences between Cases 1 and 2. For higher order systems the graphical procedure is usually followed in order to get the maximum of $|G|$ or the minimum of the ReG. In Case 2 the transfer function $G_2(j\omega)$ has the parameter q in it, and solving the problem graphically is a trial and error procedure. For a given value of q, ReG(jω) is plotted, and it must be to the right of the vertical line through -1/4q. Therefore, for different values of q, not only is the locus of G(jω) different, but the Popov line shifts also. Finding the value of q when the locus and the Popov line are tangent is a definite trial and error procedure. In Case 1 G(jω) does not have q in it and can be plotted once and its maximum value determined. Therefore, for the class of systems given above, the Bongiorno criterion has a definite advantage. For other classes of systems, one of the other criteria may have a similar advantage.

The results of the example presented here can be compared with the existing results on the Mathieu equation, to see how close the criteria of this chapter come to the necessary condition for stability. For $\zeta = .05$, the above results give $q < .05$ as being sufficient for stability. The actual stability boundary, as calculated in Phillips (1963), is $q = 0.1$ so that the sufficient condition given by the Liapunov-Popov approach is only half the actual maximum value. Better results are obtained for systems with higher damping than $\zeta = .05$.

Example 4-2

This example treats a nuclear reactor operating at some given reactivity level, and it is assumed that this reactivity level is perturbed. This perturbation is treated as a time-varying coefficient, and (4-23) is used to find a condition on the amplitude of the variation which will insure stability.

The reactor equations are (Weaver 1963)

$$\dot{n} = \frac{\delta k - \beta}{\ell} n + \sum_{i=1}^{6} \lambda_i C_i$$

$$\dot{C}_i = \frac{\beta_i}{\ell} n - \lambda_i C_i$$

The reactivity is assumed to be of the form

$$\delta k = \delta k_0 (1 + f(t))$$

The equation for $\dot{n}$ is now

$$\dot{n} = \frac{\delta k_0 - \beta}{\ell} n + \sum_{i=1}^{6} \lambda_i C_i + \frac{\delta k_0 f(t) n}{\ell}$$

The complete set of equations written in matrix form can be represented by

$$\underline{\dot{x}} = A\underline{x} + \underline{b} f(\sigma, t)$$

$$\sigma = \underline{c}' \underline{x}$$

where

$$A = \begin{bmatrix} \frac{\delta k_0 - \beta}{\ell} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \beta_1/\ell & -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \beta_2/\ell & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ \beta_3/\ell & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ \beta_4/\ell & 0 & 0 & 0 & -\lambda_4 & 0 & 0 \\ \beta_5/\ell & 0 & 0 & 0 & 0 & -\lambda_5 & 0 \\ \beta_6/\ell & 0 & 0 & 0 & 0 & 0 & -\lambda_6 \end{bmatrix}$$

and

$$\underline{b}' = \underline{c}' = \lfloor 1\ 0\ 0\ 0\ 0\ 0\ 0 \rfloor$$

In order to use the stability criterion the expression $G(s) = \underline{c}' A_s^{-1} \underline{b}$ must be calculated. It can be shown that this expression is

$$G(s) = \underline{c}' A_s^{-1} \underline{b} = \frac{1}{s + \frac{\beta - \delta k_0}{\ell} - \sum_{c=1}^{6} \frac{\lambda_i \beta_i/\ell}{s + \lambda_i}}$$

If uranium-235 is used in the reactor, the constants are (Weaver 1963) $\beta = .0064$ and

i	λ_i	β_i
1	0.0124	0.00021
2	0.0305	0.00140
3	0.111	0.00125
4	0.301	0.00253
5	1.13	0.00074
6	3.01	0.00027

The value of ℓ is 10^{-4} seconds and assume the steady state value of reactivity δk_0 is -10^{-3}.

The stability criterion is (4-23)

$$k\left|G(j\omega)\right| < 1 \tag{4-23}$$

and the maximum value of $|G(j\omega)|$ for the given constants is 0.1. Therefore $0.1k < 1$ or $k < 10$. But $k = |\delta k_0 \; f(t)|/1$. Putting the proper quantities in this expression gives $|f(t)| < 1$. Therefore the system is sure to be stable for any change of reactivity such that the new reactivity is between 0 and -2×10^{-3}.

Of course this gives a gross account of the stability region since effects such as feedback effects of the temperature on the reactivity were not considered explicitly, but were lumped together as a time-varying coefficient. Better results should be obtainable by adjoining the equations describing these effects to the above set of system equations.

4.6 Retaining the Integral of the Nonlinearity in V

In order to try to improve the sufficient conditions for stability derived earlier in this chapter, the integral of the nonlinear term is put back into the Liapunov function. The work here follows Rekasius and Rowland (1965) and is for the principal case of (2-3).

The starting point is the usual Liapunov function for the principal case

$$V = \underline{x}'P\underline{x} + \beta \int_0^{\sigma} f(z,t)dz \qquad (3\text{-}6)$$

The time derivative is

$$\dot{V} = \dot{\underline{x}}'P\underline{x} + \underline{x}'P\dot{\underline{x}} + \beta f(\sigma,t)\dot{\sigma} + \beta \int_0^{\sigma} \frac{\partial f(z,t)}{\partial t}\, dz \qquad (4\text{-}30)$$

The idea is to put bounds on $\int_0^{\sigma} \frac{\partial f}{\partial t}\, dz$ in various ways while insuring that $-\dot{V}$ is negative definite. There are three cases which can be considered.

Case I

$$\int_0^{\sigma} \frac{\partial f(z,t)}{\partial t}\, dz \leq \alpha_1 \sigma^2 \qquad (4\text{-}31)$$

Case II

$$\int_0^{\sigma} \frac{\partial f(z,t)}{\partial t}\, dz \leq \alpha_2 \sigma f(\sigma,t) \qquad (4\text{-}32)$$

Case III

$$\int_0^{\sigma} \frac{\partial f(z,t)}{\partial t}\, dz \leq \alpha_3 f(\sigma,t)^2 \qquad (4\text{-}33)$$

The three cases can be combined in different ways, but this only adds another degree of confusion to the calculations since it is not clear how much one case should be weighted compared to the others.

The equations for $-\dot{V}$ are obtained for each case by adding and subtracting $\lambda(\sigma,t)$ and the right hand side of (4-31), (4-32), and (4-33) to (4-30). The result in each case is

Case I

$$-\dot{V} = \underline{x}'(Q - \beta\alpha_1 \underline{c}\,\underline{c}')\underline{x} + \underline{x}'(2P\underline{b} - \beta A'\underline{c} - \underline{c})f(\sigma,t)$$

$$+ (\beta\underline{c}'\underline{b} + \frac{1}{k})f(\sigma,t)^2 + \lambda(\sigma,t)$$

$$+ (\beta\alpha_1\sigma^2 - \beta\int_0^\sigma \frac{\partial f(z,t)}{\partial t}\,dz) \qquad (4\text{-}34)$$

Case II

$$-\dot{V} = \underline{x}'Q\underline{x} + \underline{x}'(2P\underline{b} - \beta A'\underline{c} - \beta\alpha_2\underline{c} - \underline{c})f(\sigma,t)$$

$$+ (\beta\underline{c}'\underline{b} + \frac{1}{k})f(\sigma,t)^2 + \lambda(\sigma,t)$$

$$+ (\beta\alpha_2\sigma f(\sigma,t) - \beta\int_0^\sigma \frac{\partial f(z,t)}{\partial t}\,dt) \qquad (4\text{-}35)$$

Case III

$$-\dot{V} = \underline{x}'Q\underline{x} + \underline{x}'(2P\underline{b} - \beta A'\underline{c} - \underline{c})f(\sigma,t)$$

$$+ (\beta\underline{c}'\underline{b} - \beta\alpha_3 + \frac{1}{k})f(\sigma,t)^2 + \lambda(\sigma,t)$$

$$+ (\beta\alpha_3 f^2(\sigma,t) - \beta\int_0^\sigma \frac{\partial f(z,t)}{\partial t}\,dt) \qquad (4\text{-}36)$$

Similar to the previous development, if $-\dot{V}$ is of the form

$$-\dot{V} = (\underline{q}'\underline{x} + f(\sigma,t)/\sqrt{\tau})^2 + \underline{x}'\epsilon D\underline{x} + \lambda(\sigma,t)$$
$$+ \text{(Positive term)} \tag{4-37}$$

then by comparing (4-37) to (4-34), (4-35) and (4-36), a set of algebraic equations is obtained which can be used with Lemma 1 to obtain stability criteria. The equations are

Case I

$$\begin{aligned} Q &= \underline{q}\ \underline{q}' + \epsilon\ D + \beta\alpha_1\ \underline{c}\ \underline{c}' \\ 2\underline{q}/\sqrt{\tau} &= 2P\underline{b} - \beta A'\underline{c} - \underline{c} \\ \frac{1}{\tau} &= \beta\underline{c}'\underline{b} + \frac{1}{k} \end{aligned} \tag{4-38}$$

Case II

$$\begin{aligned} Q &= \underline{q}\ \underline{q}' + \epsilon\ D \\ 2\underline{q}/\sqrt{\tau} &= 2P\underline{b} - \beta A'\underline{c} - \underline{c} - \beta\alpha_2\underline{c} \\ \frac{1}{\tau} &= \beta\underline{c}'\underline{b} + \frac{1}{k} \end{aligned} \tag{4-39}$$

Case III

$$\begin{aligned} Q &= \underline{q}\ \underline{q}' + \epsilon\ D \\ 2\underline{q}/\sqrt{\tau} &= 2P\underline{b} - \beta A'\underline{c} - \underline{c} \\ \frac{1}{\tau} &= \beta\underline{c}'\underline{b} - \beta\alpha_3 + \frac{1}{k} \end{aligned} \tag{4-40}$$

The application of Lemma 1 to these equations results in the following stability criteria.

Case I

$$\frac{1}{k} + \mathrm{Re}(1 + j\omega\beta)G(j\omega) - \beta\alpha_1|G(j\omega)|^2 > 0 \tag{4-41}$$

Case II

$$\frac{1}{k} + \mathrm{Re}(1 + \beta\alpha_2 + j\beta\omega)G(j\omega) > 0 \tag{4-42}$$

Case III

$$\frac{1}{k} - \beta\alpha_3 + \mathrm{Re}(1 + j\omega\beta)G(j\omega) > 0 \tag{4-43}$$

Therefore, if some β can be found such that one of the inequalities (4-41), (4-42), or (4-43) holds for all real ω, then the system (2-3) is absolutely stable.

Since the Bongiorno type of criterion had an advantage for certain systems in the above work, the analogous case was also investigated here. This advantage did not carry over however, and the results are more complicated than those given above. Therefore, they are not included here.

An example is now given which illustrates when the above criteria give improved results over the previous case (4-7). The same equation as in Example 4-1 is considered.

Example 4-3

The equation is

$$\ddot{x} + 2\zeta\dot{x} + (1 - 2q \cos 2t)x = 0$$

In matrix form this is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1 - 2q) & -2\zeta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2q(1 - \cos 2t)x_1$$

To use the above criteria, the numbers α_1, α_2, and α_3 must be calculated.

$$f(x_1,t) = 2q(1 - \cos 2t)x_1$$

$$0 \leq f(x_1,t)/x_1 \leq 4q$$

$$\frac{\partial f(x_1,t)}{\partial t} = 4q(\sin 2t)x_1$$

$$\int_0^{x_1} \frac{\partial f(z,t)}{\partial t}\, dz = 2q(\sin 2t)x_1^2$$

For the three cases (4-31), (4-32), and (4-33), the inequalities are

Case I

$$2q(\sin 2t)x_1^2 \leq \alpha_1 x_1^2$$

$$\alpha_1 = 2q$$

Case II

$$2q(\sin 2t)x_1^2 \leq \alpha_2 x_1 2q(1 - \cos 2t)x_1$$

$$\alpha_2 = \max \frac{\sin 2t}{1 - \cos 2t} = \infty$$

Case III

$$2q(\sin 2t)x_1^2 = \alpha_3 4q^2(1 - \cos 2t)^2 x_1^2$$

$$\alpha_3 = \max \frac{\sin 2t}{2q(1 - \cos 2t)^2} = \infty$$

The stability criterion for Case III, (4-43), can only be true for $\alpha_3 = \infty$ if $\beta = 0$. Therefore this criterion reduces to the previous case, (4-7). In Case II $\alpha_2 = \infty$ means that (4-42) becomes

$$\mathrm{Re}\,\beta\alpha_2 G(j\omega) > 0$$

The transfer function $G(j\omega)$ is

$$G(j\omega) = \frac{1}{1 - 2q - \omega^2 + j2\omega}$$

$$\mathrm{Re}G(j\omega) = \frac{1 - 2q - \omega^2}{(1 - 2q - \omega^2)^2 + 4\zeta^2\omega^2}$$

The sign of $\mathrm{Re}G(j\omega)$ changes as ω goes from zero to infinity so that again this criterion is no good unless $\beta = 0$.

This just leaves Case I. The criterion in this case is

$$\frac{1}{k} + \mathrm{Re}(1 + j\omega\beta)G(j\omega) - \beta 2q|G(j\omega)|^2 > 0$$

The quantities $|G(j\omega)|^2$ and $\mathrm{Re}(1 + j\beta\omega)G(j\omega)$ are

$$|G(j\omega)|^2 = \frac{1}{(1 - 2q - \omega^2)^2 + 4\zeta^2\omega^2}$$

$$\mathrm{Re}(1 + j\beta\omega)G(j\omega) = \frac{1 - 2q - \omega^2 + 2\beta\zeta\omega^2}{(1 - 2q - \omega^2)^2 + 4\zeta^2\omega^2}$$

so that the criterion is

$$\frac{1}{4q} + \frac{1 - 2q - \omega^2 + 2\beta\zeta\omega^2 - 2q\beta}{(1 - 2q - \omega^2)^2 + 4\zeta^2\omega^2} > 0$$

Multiplying thorough by $4q((1 - 2q - \omega^2)^2 + 4\zeta^2\omega^2)$ gives

$$(1 - 2q - \omega^2)^2 + 4\zeta^2\omega^2 + 4q(1 - 2q - \omega^2)$$

$$+ 8\beta\omega^2\zeta q - 8\beta q^2 > 0$$

This can be rearranged as

$$\omega^4 + \omega^2(-2 + 4\zeta^2 + 8\beta\zeta q) + 1 - 4q^2 - 8\beta q^2 > 0$$

Finding the frequency at which this is minimum gives $\omega^2 = 1 - 2\zeta^2 - 4\beta\zeta q$ or, when $1 - 2\zeta^2 - 4\beta\zeta q$ is negative, $\omega = 0$. For the case where $\omega = 0$ the criterion is

$$1 - 4q^2 - 8\beta q^2 > 0$$

or

$$q^2 = \frac{1}{4 + 8\beta}$$

Therefore q is maximum when $\beta = 0$ and the maximum $q = 0.5$ for $\zeta > .707$.

In the other case substituting for ω^2 leads to the inequality

$$-(1 - 2\zeta^2 - 4\beta\zeta q)^2 + 1 - 4q^2 - 8\beta q^2 > 0$$

From the previous example, the maximum value of q using the Popov criterion was $q < 0.05$ for $\zeta = 0.05$. If these values are substituted into the above expression then it can be seen that again β must be zero for the above expression to be satisfied unless q is made smaller than 0.05.

Therefore, the inclusion of the integral term into the Liapunov function is no help at all for the equation under study. One reason for this is that the frequency of the time variation is at a critical value. This equation is the damped Mathieu equation which has been studied, by other means, by McLachlan (1953) and Cunningham (1954). One of the results of these studies is that, if the frequency of the cosine term is twice the natural frequency of the constant part of the equation, then this is a critical value as far as stability is concerned. This holds true for all higher even harmonics of the natural frequency. Therefore, the stability does not depend on the rate of variation directly, but on the relation of the rate of variation to the natural frequency of the equation.

This brings up the question of whether the frequency criteria of (4-41)-(4-43) are any good at all. The answer is that these criteria should be applicable whenever the frequency of the variation is less than twice the resonant frequency of the equation. In higher order cases this should hold if the frequency of the variation is less than twice the natural frequency of any dominant complex roots. This is just a conjecture, however.

Case I of the above problem is now reworked with the frequency of the variation decreased by half. Everything is the same as before except that cos 2t is replaced by cos t, which then changes the value of α_1 to $\alpha_1 = q$. The stability criterion is

$$\frac{1}{4q} + \mathrm{Re}(1 + j\beta\omega)G(j\omega) - q\beta\left|G(j\omega\right|^2 > 0$$

or substituting the transfer function into this inequality gives

$$\frac{1}{4q} + \frac{1 - 2q - \omega^2 + 2\beta\zeta\omega^2 - q\beta}{(1 - 2q - \omega^2)^2 + 4\zeta^2\omega^2} > 0$$

Manipulating this expression gives

$$\omega^4 + \omega^2(-2 + 4\zeta^2 + 8\beta\zeta q) + 1 - 4q^2 - 4\beta q^2 > 0$$

which is minimum for $\omega^2 = 1 - 2\zeta^2 - 4\beta\zeta q$. Substituting this into the equation gives

$$-(1 - 2\zeta^2 - 4\beta\zeta q)^2 + 1 - 4q^2 - 4\beta q^2 > 0$$

Let $\zeta = .05$. The inequality becomes

$$-.04\beta^2 q^2 + \beta(.398q - 4q^2) + .01 - 4q^2 > 0$$

It can be shown that the maximum value of q which satisfies this inequality is $q = .0856$ when $\beta = 7.93$. Therefore, the sufficient condition for stability is $q < .0856$ which is an improvement over the previous result of $q < .05$. Therefore, the inclusion of the integral term into the Liapunov function does lead to an improvement in the stability criteria if the time variations are slow enough.

This chapter developed the Popov criterion for time-varying systems, and showed how the Popov criterion is equivalent to the work of Bongiorno, Sandberg, and Narendra and Goldwyn. The basic Popov criterion was then extended by following the work of Rekasius and Rowland, and this extension was shown to give improved results when the time variations are sufficiently slow.

Now that stability criteria have been developed for nonlinear and/or time-varying systems with one nonlinear and/or time-varying element, the case of many such elements is considered. This is done in the next chapter.

Chapter 5

SYSTEMS WITH MANY NONLINEAR AND/OR TIME-VARYING ELEMENTS

5.1 Introduction

This chapter contains extensions of the previous work to systems with more than one nonlinear or time-varying element. These are the types of systems which are described by equations of the form (2-1). Obtaining stability criteria for these systems comprises most of the original contributions of this work.

The second section starts with the principal case of the system with m nonlinear elements. A Liapunov function, analogous to the one used in Chapter 3, is used, and the result is a set of algebraic equations, which must have a solution, if V and $-\dot{V}$ are to be positive definite. This leads to an extension of the matrix-inequality method so that it can be used for systems with more than one nonlinearity. A new lemma, which is a generalization of Lemma 1, is proven. This lemma is then used to get the stability criterion, the result being that a matrix which is a function of real frequency must be positive definite. For one nonlinearity this reduces to the previous work. A comparison of the stability criterion with previous work in this area is made, and three examples are worked illustrating the various features and short-comings of the criterion.

In the third section the criterion is extended to time-varying systems. Results, completely analogous to those obtained in Chapter 4, are obtained. The application of the criteria is illustrated by means of an example.

The fourth section contains a discussion of the particular case. It is shown that, in general, the particular case cannot be extended for the systems with many nonlinearities. However, one particular class of systems which does permit an extension is given, and a stability criterion is derived and its use illustrated by an example. The last section contains conclusions.

5.2 Multiple Nonlinearities--Principal Case

In this section the previous results are extended to obtain a frequency domain criterion for the principal case of the system with m nonlinear elements. The time invariant case is considered in this section, while the time-varying nonlinear case is considered in the next.

The system equations are given by (2-1) and are repeated here for convenience.

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} - B\,\underline{f}(\underline{\sigma}) \\ \underline{f}(\underline{\sigma})' &= \lfloor f_1(\sigma_1)\ f_2(\sigma_2)\ \ldots\ f_m(\sigma_m) \rfloor \\ \underline{\sigma} &= C'\underline{x}, \qquad 0 \le f_i(\sigma_i)/\sigma_i \le k_i\ , \qquad i = 1, \ldots, m \end{aligned} \tag{5-1}$$

By analogy with the previous work, a Liapunov function is chosen to be of the form

$$V = \underline{x}'P\underline{x} + \int_0^{\underline{\sigma}} \underline{f}(\underline{z})'\overline{\beta}\, d\underline{z} \tag{5-2}$$

where $\overline{\beta} = \mathrm{diag}\ (\beta_1, \beta_2, \ldots, \beta_m)$. Taking the time derivative of (5-2) gives

$$\dot{V} = \underline{x}'(A'P + PA)\underline{x} - 2\underline{x}'PB\underline{f}(\underline{\sigma}) + \underline{f}(\underline{\sigma})'\overline{\beta}\ \dot{\underline{\sigma}} \tag{5-3}$$

Substituting for $\dot{\underline{\sigma}}$ gives

$$\dot{V} = \underline{x}'(A'P + PA)\underline{x} - 2\underline{x}'PB\underline{f}(\underline{\sigma}) + \underline{f}(\underline{\sigma})'\overline{\beta}(C'A\underline{x} - C'B\underline{f}(\underline{\sigma}))$$

Writing $-\dot{V}$ and adding and subtracting the term $\underline{f}'(\underline{\sigma})(\underline{\sigma} - K^{-1}\underline{f}(\underline{\sigma}))$ gives

$$-\dot{V} = \underline{x}'Q\underline{x} + \underline{x}'(2PB - A'C\overline{\beta})\underline{f}(\underline{\sigma})$$

$$+ \underline{f}(\underline{\sigma})'\overline{\beta}C'B\underline{f}(\underline{\sigma}) \pm \underline{f}(\underline{\sigma})'\ (\underline{\sigma} - K^{-1}\underline{f}(\underline{\sigma})) \tag{5-4}$$

where $K = \mathrm{diag}\ (k_1, k_2, \ldots, k_m)$. Letting $\underline{f}(\underline{\sigma})'(\underline{\sigma} + K^{-1}\underline{f}(\underline{\sigma})) = \lambda(\underline{\sigma})$ and rewriting (5-4) gives

$$-\dot{V} = \underline{x}'Q\underline{x} + \underline{x}'(2PB - A'C\overline{\beta} - C)\ \underline{f}(\underline{\sigma})$$

$$+ \underline{f}(\underline{\sigma})'(\tfrac{1}{2}(\overline{\beta}C'B + B'C\overline{\beta}) + K^{-1})\ \underline{f}(\underline{\sigma})$$

$$+ \lambda(\underline{\sigma}) \tag{5-5}$$

The symmetric part of the quadratic form $\underline{f}'\overline{\beta}C'B\underline{f}$ is used, since a quadratic form is completely specified by its symmetric part.

In a manner completely analogous to the previous work, it is desired to put $-\dot{V}$ in the form (Sultanov 1964)

$$-\dot{V} = (Q_2'\underline{x} + T\underline{f}(\underline{\sigma}))'\ (Q_2'\underline{x} + T\underline{f}(\underline{\sigma}))$$

$$+ \underline{x}'\ \epsilon D\underline{x} + \lambda(\underline{\sigma}) \tag{5-6}$$

where Q_2 is an n by m matrix. Multiplying this out gives

$$-\dot{V} = \underline{x}'Q_2Q_2'\underline{x} + 2\underline{x}'Q_2T\underline{f}(\underline{\sigma}) + \underline{f}(\underline{\sigma})'T'T\underline{f}(\underline{\sigma})$$

$$+ \underline{x}' \in D\underline{x} + \lambda(\underline{\sigma}) \tag{5-7}$$

Comparing (5-7) and (5-5) results in the following set of equations

$$Q = Q_2Q_2' + \epsilon D = -(A'P + PA) \tag{5-8}$$

$$2Q_2T = 2P\bar{B} - A'C\bar{\beta} - \bar{C} \tag{5-9}$$

$$T'T = \frac{1}{2}(\bar{\beta}C'B + B'C\bar{\beta}) + K^{-1} \tag{5-10}$$

In order for the first three terms in (5-5) to be a positive definite form in $\underline{x}$ and $\underline{f}$, it is necessary, but not sufficient, for the matrix $T'T$ to be positive definite. Therefore, if (5-8) and (5-9) can be solved for Q_2 and $P > 0$, then V and $-\dot{V}$ exist and are positive definite, and the system (5-1) is absolutely stable. The conditions for the existence of the solution of (5-8) and (5-9) can be found with the help of a lemma, which is an extension of Lemma 1 of Chapter 3. The statement and proof of this lemma follows next.

Lemma 2: Given a stable, n by n, real matrix A; symmetric, n by n, real matrices $D > 0$ and $R \geq 0$; n by m, real matrices G of rank m and $H \neq 0$; an m by m, real matrix $T'T > 0$; and scalars $\epsilon > 0$ and ρ, where ρ is such that the right hand side of equation (aa) below is negative definite; then a necessary and sufficient condition for the solution as a symmetric, n by n, real matrix $P > 0$ and an n by m, real matrix Q_2 of the system

$$PA + A'P = -Q_2Q_2' - \epsilon D - \rho R \tag{aa}$$

$$Q_2T = PG - H \tag{bb}$$

is that ϵ be small enough and that the matrix

$$T'T + 2\mathrm{He}H'A_\omega^{-1}G - \rho G'A_\omega^{-1*} RA_\omega^{-1} G \qquad (**)$$

is positive definite for all real ω.

The notation HeM means the Hermitian part of the matrix M.

Proof of Necessity

The proof again starts by using the identity

$$A'P + PA = -(PA_\omega + A_\omega^* P) \qquad (5\text{-}11)$$

in (aa). This substitution results in the equation

$$PA_\omega + A_\omega^* P = Q_2Q_2' + \rho R + \epsilon D \qquad (5\text{-}12)$$

(5-12) is premultiplied by $G'A_\omega^{-1*}$ and post-multiplied by $A_\omega^{-1}G$ giving

$$G'A_\omega^{-1*} PG + G'PA_\omega^{-1}G = G'A_\omega^{-1*} Q_2Q_2'A_\omega^{-1}G$$
$$+ \rho G'A_\omega^{-1*} RA_\omega^{-1}G + \epsilon G'A_\omega^{-1*} DA_\omega^{-1}G \qquad (5\text{-}13)$$

Using (bb) to substitute for PG gives

$$G'A_\omega^{-1*} Q_2T + G'A_\omega^{-1*} H + T'Q_2'A_\omega^{-1}G$$
$$+ H'A_\omega^{-1}G = G'A_\omega^{-1*} Q_2Q_2'A_\omega^{-1}G + \rho G'A_\omega^{-1*} RA_\omega^{-1}G$$
$$+ \epsilon G'A_\omega^{-1*} DA_\omega^{-1}G \qquad (5\text{-}14)$$

Rearranging terms gives

$$2HeH'A_\omega^{-1}G - \rho G'A_\omega^{-1*} RA_\omega^{-1}G$$

$$= G'A_\omega^{-1*} Q_2Q_2'A_\omega^{-1}G - 2HeT'Q_2'A_\omega^{-1}G + \Delta \qquad (5\text{-}15)$$

The term $\Delta = \epsilon G'A_\omega^{-1*} DA_\omega^{-1}G$ is positive definite. To see that this is true, first recognize that, as before, $D_1 = A_\omega^{-1*} DA_\omega^{-1}$ is positive definite since D is positive definite. Since D_1 is positive definite, it can be written as the product of two nonsingular matrices $E'E$ so that $G'D_1G = G'E'EG$. E is an n by n nonsingular matrix and G is an n by m matrix of rank $m \leq n$. Therefore, the vector $\underline{x} = EG\underline{y}$ is only zero if $\underline{y}$ is zero, and $\underline{x}'\underline{x} = \underline{y}'G'E'EG\underline{y}$ is greater than zero for all $\underline{y} \neq \underline{0}$ and equals zero only for $\underline{y} = \underline{0}$, so that $G'E'EG$ is positive definite. Therefore, $\epsilon G'A_\omega^{-1*} DA_\omega^{-1}G = \Delta$ is positive definite.

Adding $T'T$ to both sides of (5-15) leads to

$$T'T + 2HeH'A_\omega^{-1}G - \rho\, G'A_\omega^{-1*} RA_\omega^{-1}G$$

$$= (Q_2'A_\omega^{-1}G - T)^* \; (Q_2'A_\omega^{-1}G - T) + \Delta \qquad (5\text{-}16)$$

But if A is a complex matrix, then A^*A is at least a non-negative definite Hermitian matrix. Therefore, the right side of (5-16) is positive definite so that the left side must also be positive definite. The left side is just (**) so that the necessity of that condition is proved.

Proof of Sufficiency

The matrices $T'T + 2HeH'A_\omega^{-1}G - \rho G'A_\omega^{-1*} RA_\omega^{-1}G$ and $G'A^{-1*} DA^{-1}G$ are positive definite for all ω. The value of ϵ can always be chosen small enough such that the matrix $T'T + 2HeH'A_\omega^{-1}G - \rho G'A_\omega^{*-1}RA_\omega^{-1}G - \epsilon G'A_\omega^{*-1}DA_\omega^{-1}G$ is also positive definite.

Let $a(s) = \det A_s$ which is a real polynomial with leading coefficient unity. The elements of the last three terms of the above matrix all have the term $1/a(j\omega)a(-j\omega)$ in them, coming from the A_ω^{-1} and A_ω^{-1*} terms. The above matrix is also positive definite and Hermitian, and, as indicated above, it can therefore be written as the product of a complex matrix and the adjoint of that complex matrix. Therefore, the matrix must take the following form

$$T'T + 2HeH'A_\omega^{-1}G - \rho G'A_\omega^{*-1}RA_\omega^{-1}G - \epsilon G'A_\omega^{-1*} DA_\omega^{-1}G$$

$$= (T + \frac{1}{a(j\omega)} V(j\omega))^* (T + \frac{1}{a(j\omega)} V(j\omega)) \qquad (5\text{-}17)$$

By analogy with the previous work (i.e., **Lemma 1**), the matrix $\frac{1}{a(j\omega)} V(j\omega)$ is set equal to $-Q_2'A_\omega^{-1}G$. This leads to a set of equations which can be solved for the elements of Q_2. Once Q_2 is known it can be used in (aa) to find P, which is positive definite by Liapunov's theorem.

By going back to the necessity proof, it can be shown that Q_2, defined as above, satisfies (bb). Substituting the expression $-Q_2'A_\omega^{-1}G$ into (5-17) gives

$$T'T + 2HeH'A_\omega^{-1}G - \rho G'A_\omega^{-1*}RA_\omega^{-1}G - \epsilon G'A_\omega^{-1*} DA_\omega^{-1}G$$

$$= (T - Q_2'A_\omega^{-1}G)^* (T - Q_2'A_\omega^{-1}G) \qquad (5\text{-}18)$$

Performing the multiplication on the right hand side and cancelling the T'T terms gives

$$2HeH'A_\omega^{-1}G - \rho G'A_\omega^{-1*} RA_\omega^{-1}G - \epsilon G'A_\omega^{-1*} DA_\omega^{-1}G$$

$$= G'A_\omega^{-1*} Q_2Q_2'A_\omega^{-1}G - 2HeT'Q_2'A_\omega^{-1}G \qquad (5\text{-}19)$$

Using (5-13) in (5-19) gives

$$2HeH'A_\omega^{-1}G - \rho H'A_\omega^{-1*} RA_\omega^{-1}G - \epsilon G'A_\omega^{-1*} DA_\omega^{-1}G$$

$$= G'A_\omega^{-1*} PG + G'PA_\omega^{-1}G - \rho G'A_\omega^{-1*} RA_\omega^{-1}G$$

$$- \epsilon G'A_\omega^{-1*} DA_\omega^{-1}G - 2HeT'Q_2'A_\omega^{-1}G \qquad (5\text{-}20)$$

Cancelling the proper terms and rearranging the equation gives

$$2HeH'A_\omega^{-1}G + 2HeT'Q_2'A_\omega^{-1}G - 2HeG'PA_\omega^{-1}G = 0$$

or

$$2He\ (H' + T'Q_2' - G'P)\ A_\omega^{-1}G = 0 \qquad (5\text{-}21)$$

Since $A_\omega^{-1}G \neq 0$ and has rank m, (5-21) can only be true if

$$H' + T'Q_2' - G'P = 0$$

Or, by taking the transpose and rearranging,

$$Q_2T = PG - H \qquad \textbf{(bb)}$$

Therefore, the Q_2 matrix satisfies (bb), and a solution Q_2 and P to (aa) and (bb) has been found using (**), so that this condition is sufficient.

Now that the lemma has been proved, equations (5-8)-(5-10) can be considered again. Comparing (5-9) with (bb) and repeating (5-10) gives the equations

$$G = B \qquad \rho = 0$$

$$H = \frac{1}{2}(A'C\bar{\beta} + C) \tag{5-22}$$

$$T'T = \frac{1}{2}(\bar{\beta}C'B + B'C\bar{\beta}) + K^{-1}$$

Substituting (5-22) into (**) gives the condition for (5-8) and (5-9) to have a solution Q_2 and $P > 0$. This is

$$\frac{1}{2}(\bar{\beta}C'B + B'C\bar{\beta}) + K^{-1} + \text{He}(\bar{\beta}C'AA_{\omega}{}^{-1}B + C'A_{\omega}{}^{-1}B) > 0 \tag{5-23}$$

But

$$A = j\omega I - A_{\omega}$$

and

$$AA_{\omega}{}^{-1} = j\omega A_{\omega}{}^{-1} - I$$

Substituting this equation into (5-23) gives

$$\frac{1}{2}(\bar{\beta}C'B + B'C\bar{\beta}) + K^{-1} + \text{He}(\bar{\beta}C'j\omega A_{\omega}{}^{-1}B - \bar{\beta}C'B + C'A_{\omega}{}^{-1}B) > 0 \tag{5-24}$$

But $\text{He}(-\bar{\beta}C'B) = -\frac{1}{2}(\bar{\beta}C'B + B'C\bar{\beta})$ so that the final result is

$$K^{-1} + \text{He}(I + j\omega\bar{\beta})C'A_{\omega}{}^{-1}B > 0 \tag{5-25}$$

Therefore, given $C'A_{\omega}{}^{-1}B$ and K, if a $\bar{\beta} = \text{diag}\ (\beta_1, \ldots, \beta_m)$ can be found such that the matrix on the left in (5-25) is positive definite, then the system (5-1) is absolutely stable.

Other work on this type of system of equations has been done by Popov (1960) and Ibrahim and Rekasius (1964). Their results are essentially the same as those obtained here, but the method of derivation is considerably different. Their main theorems are presented here.

Theorem of Popov: If, being given the system (5-1), with A stable, one is able to find three diagonal matrices P, Q, K possessing the following properties:

1. The diagonal elements p_i and k_i of P and K are positive
2. The Hermitian matrix

$$H(\omega) = \frac{1}{2}(G(\omega) + G^*(\omega)) \tag{5-26}$$

where

$$G(\omega) = -(P + j\omega Q)C'(j\omega I - A)^{-1}B + PK^{-1} \tag{5-27}$$

and where $G^*(\omega)$ is the adjoint of the matrix $G(\omega)$, satisfied Sylvester's conditions (that is to say is strictly positive definite) whatever the real number ω.

3. The symmetric matrix

$$S = -\frac{1}{2}QC'B - \frac{1}{2}(QC'B)' + PK^{-1} \tag{5-28}$$

where (QCB)' is the transpose of the matrix QC'B, satisfies Sylvester's conditions.

From these conditions, the trivial solution of the system (5-1) is asymptotically stable in the large whatever the function $\underline{f}(\underline{y})$, whose components $f_i(y_i)$ satisfy the inequality

$$0 \leq f_i(y_i)y_i \leq k_i y_i^2$$

This theorem can be made to look exactly like the results derived above by premultiplying (5-27) and (5-28) by P^{-1} and letting $P^{-1}Q = \overline{\beta}$. Then (5-26) is the same as (5-25) and (5-28) is the same as (5-10).

Theorem of Ibrahim and Rekasius: The system (5-1) is globally asymptotically stable if there exists a non-negative diagonal matrix Q, and two positive diagonal matrices G and H such that

1. $-\underline{f}(\underline{\sigma})'QC'B\underline{f}(\underline{\sigma}) \leq 0$, $\underline{f}(\underline{\sigma}) \neq 0$
2. the elements of GH^{-1} satisfy the inequality

$$0 < \sigma_i f(\sigma_i) \leq g_i\sigma_i^2/k$$

3. the matrix inequality

$$T(j\omega) = H + \frac{1}{2}(QC'A + GC')(j\omega I - A)^{-1}B$$

$$+ B'(-j\omega I - A')^{-1} (QC'A + GC')') > 0 \tag{5-29}$$

holds for all real ω.

4. A is asymptotically stable.

Note that the B matrix here has the opposite sign of Ibrahim and Rekasius' paper. This theorem requires some manipulation before it can be compared with the previously derived results. Rewriting (5-29) gives

$$H + \text{He } (QC'A + GC')A_\omega^{-1}B > 0$$

$$H + \text{He } (QC'AA_\omega^{-1}B + GC'A_\omega^{-1}B) > 0$$

But $AA_\omega^{-1} = j\omega A_\omega^{-1} - I$. Therefore

$$H + \text{He}(j\omega QC'A_\omega^{-1}B - QC'B + GC'A_\omega^{-1}B) > 0$$

$$H + \text{He}((j\omega Q + G)C'A_\omega^{-1}B - QC'B) > 0$$

Taking out the QC'B term and letting G = I gives

$$H - \text{He}QC'B + \text{He}(I + j\omega Q)C'A_\omega^{-1}B > 0 \tag{5-30}$$

Letting $H = K^{-1}$ and $Q = \overline{\beta}$ gives the same notation as the previous work. If (5-30) is compared with (5-25), it is seen that there is an extra term present which is not in (5-25). This term is required to be negative semidefinite so that it makes (5-30) a more restrictive criterion than (5-25). The reason is that the first requirement of the theorem is that $-\underline{f}'QC'B\underline{f}$ be less than or equal to zero. This is a condition which is not required by (5-25). The analogous condition in the development of (5-25) is (5-10), that is,

$$\frac{1}{2}(\bar{\beta}C'B + B'C\bar{\beta}) + K^{-1} > 0 \tag{5-10}$$

This inequality can hold true even if $\bar{\beta}C'B + B'C\bar{\beta}$ is not positive semi-definite. Since $\bar{\beta} = Q$, then $QC'B + (QC'B)'$ is not required to be positive semidefinite. An example of a case where the first condition of Ibrahim and Rekasius' theorem is violated, so that their theorem cannot be applied, is given later. However, (5-25) is able to give results in this case.

Actually, the criterion of Ibrahim and Rekasius can be derived from Lemma 2 by writing $\dot{V}$ as

$$-\dot{V} = (Q_2'\underline{x} + T\underline{f}(\underline{\sigma}))'(Q_2'\underline{x} + T\underline{f}(\underline{\sigma}))$$

$$+ \underline{x}'\epsilon D\underline{x} + \underline{f}(\underline{\sigma})'\bar{\beta}C'B\,\underline{f}(\underline{\sigma}) + \lambda(\underline{\sigma}).$$

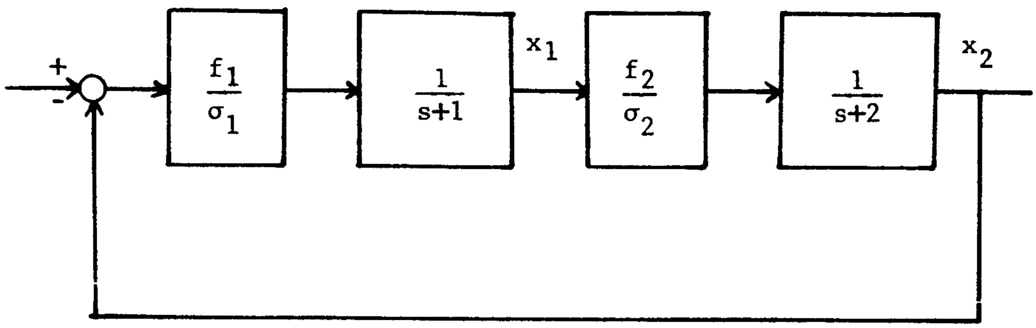
This does not change the G and H matrices in (5-22), but now $T'T = K^{-1}$. Therefore, applying Lemma 2, with the additional restriction that $\underline{f}'\bar{\beta}C'B\underline{f}$ is positive semidefinite, results in 5-30.

Example 5-1

Consider the system whose block diagram and equations are given in Fig. 6. The stability criterion given by (5-25) is

$$K^{-1} + \mathrm{He}(I + j\omega\bar{\beta})C'A_\omega^{-1}B > 0 \tag{5-25}$$

Calculating $C'A_\omega^{-1}B$ gives



a) Block Diagram Defining the State Variables

$$\dot{\underline{x}} = A\underline{x} - B\underline{f}(\underline{\sigma}), \quad \underline{\sigma} = C'\underline{x}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad C' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

b) System Equations

Fig. 6. System of Example 5-1.

$$C'A_\omega^{-1}B = \begin{bmatrix} 0 & \frac{1}{j\omega+2} \\ \frac{-1}{j\omega+1} & 0 \end{bmatrix}$$

$$(I + j\omega\bar{\beta})C'A_\omega^{-1}B = \begin{bmatrix} 0 & \frac{1+j\omega\beta_1}{j\omega+2} \\ -\frac{1+j\omega\beta_2}{j\omega+1} & 0 \end{bmatrix}$$

Substituting this into (5-25) gives the following as the stability criterion.

$$\begin{bmatrix} \frac{1}{k_1} & \frac{1}{2}\left(\frac{1+j\omega\beta_1}{2+j\omega} - \frac{1-j\omega\beta_2}{1-j\omega}\right) \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} \end{bmatrix} > 0$$

or

$$\begin{bmatrix} \frac{1}{k_1} & \frac{1}{2}\,\frac{(\beta_1 - \beta_2)\omega^2 + (\beta_1 - 2 + 2\beta_2)j\omega - 1}{(2 + j\omega)(1 - j\omega)} \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} \end{bmatrix} > 0$$

Setting the coefficient of ω^2 and $j\omega$ in the numerator of the frequency dependent element to zero gives

$$\beta_1 - \beta_2 = 0, \; \beta_1 = \beta_2$$

$$\beta_1 - 2 + 2\beta_2 = 0, \quad \beta_1 = \frac{2}{3} = \beta_2$$

The matrix is now

$$\begin{bmatrix} \frac{1}{k_1} & \frac{-1}{2(2 + j\omega)(1 - j\omega)} \\ \frac{-1}{2(2 - j\omega)(1 + j\omega)} & \frac{1}{k_2} \end{bmatrix} > 0$$

Applying Sylvester's condition gives

$$\frac{1}{k_1 k_2} - \frac{1}{4(4 + \omega^2)(1 + \omega^2)} > 0$$

The frequency dependent term has its largest magnitude at $\omega = 0$ so that $k_1 k_2 < 16$ is sufficient for the system to be stable.

Actually a much better answer can be obtained from a Liapunov function which is just the sum of the integrals of the two nonlinearities with $\beta_1 = \beta_2$. If $\beta_1 = 1$, then in this case

$$\dot{V} = -2f_1(\sigma_1)\sigma_1 - f_2(\sigma_2)\sigma_2$$

and the system is stable for all nonlinearities which lie in the first and third quadrants. This result cannot be generalized since this type of V-function usually results in indefinite $\dot{V}$-functions.

If $\beta_1 = \beta_2 = 1$ is put into the stability criterion matrix, instead of $\beta_1 = \beta_2 = \frac{2}{3}$, the result is

$$\begin{bmatrix} \frac{1}{k_1} & \frac{-1}{2(2 + j\omega)} \\ \\ \frac{-1}{2(2 - j\omega)} & \frac{1}{k_2} \end{bmatrix} > 0$$

or

$$\frac{1}{k_1 k_2} - \frac{1}{4(4 + \omega^2)} > 0$$

or $k_1 k_2 < 16$ as before. It has been seen that, by making $P = 0$, an infinitely better result is obtained. Why doesn't this result appear from the stability criterion? The answer to this question is obtained by looking at (5-8), (5-9) and (5-22). The stability criterion gives necessary and sufficient conditions for the solution of (5-8) and (5-9). By making P and Q_2 zero, equation (5-9) becomes $H = - A'C\bar{\beta} - C = 0$. This is never true for $\beta_1 = \beta_2 = 1$ so that $P = 0$ is not a solution to the set of equations.

Example 5-2

Consider the same system as Ibrahim and Rekasius; the block diagram and equations are given in Fig. 7. Again the stability criterion is given by (5-25).

$$K^{-1} + He(I + j\omega\beta)C'A_\omega^{-1}B > 0 \tag{5-25}$$

The term A_ω^{-1} is

$$A_\omega^{-1} = \frac{1}{(j\omega+5)(j\omega+3)(j\omega+2)}\begin{bmatrix} (j\omega+3)(j\omega+2) & 0 & 0 \\ 0 & (j\omega+5)^2 & (j\omega+5) \\ 0 & -6(j\omega+5) & j\omega(j\omega+5) \end{bmatrix}$$

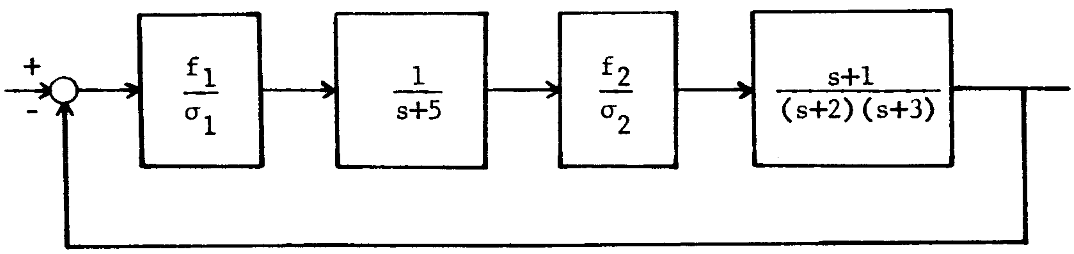
Calculating $C'A_\omega^{-1}B$ gives

$$C'A_\omega^{-1}B = \begin{bmatrix} 0 & \dfrac{1+j\omega}{(j\omega+3)(j\omega+2)} \\ -\dfrac{1}{j\omega+5} & 0 \end{bmatrix}$$

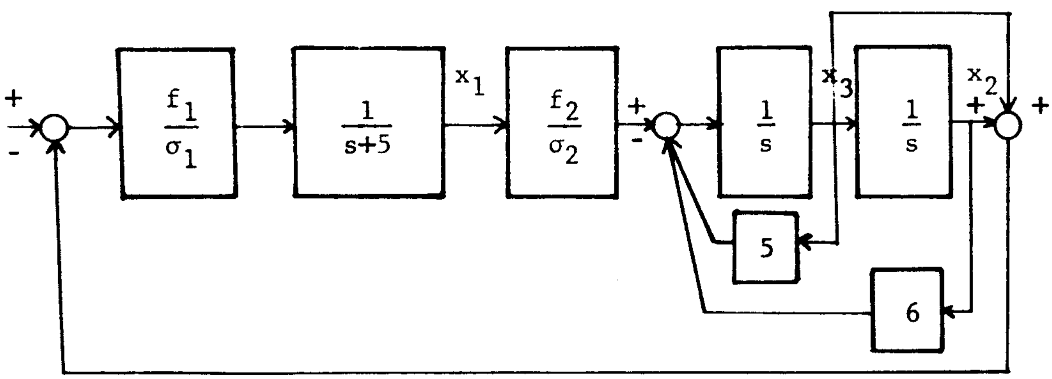
This equation is multiplied by $(j\omega\overline{\beta} + I)$ giving

$$(j\omega\overline{\beta} + I)C'A_\omega^{-1}B = \begin{bmatrix} 0 & \dfrac{(1+j\omega)(1+j\omega\beta_1)}{(j\omega+3)(j\omega+2)} \\ \dfrac{-j\omega\beta_2+1}{j\omega+5} & 0 \end{bmatrix}$$

Putting this quantity into (5-25) results in



a) System of Ibrahim and Rekasius.



b) Block Diagram Defining the State Variables.

$$\dot{\underline{x}} = A\underline{x} - B\underline{f}, \quad \underline{\sigma} = C'\underline{x}$$

$$A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad C' = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

c) System Equations

Fig. 7. System of Example 5-2.

$$\begin{bmatrix} \frac{1}{k_1} & \frac{1}{2}\frac{(1+j\omega)(1+j\omega\beta_1)}{(j\omega+3)(j\omega+2)} - \frac{1-j\omega\beta_2}{5-j\omega} \\ \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} \end{bmatrix} > 0$$

This can be rewritten as

$$\begin{bmatrix} \frac{1}{k_1} & \frac{1}{2}\frac{(\beta_1-\beta_2)j\omega^3+(2-4\beta_1-5\beta_2)\omega^2+(5\beta_1+6\beta_2-1)j\omega-1}{(j\omega+3)(j\omega+2)(5-j\omega)} \\ \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} \end{bmatrix} > 0$$

The matrix $C'B$ in this example is skew symmetric so that if $\beta_1 = \beta_2$, then the expression $\bar{\beta}C'B + B'C\bar{\beta} = 0$. The Ibrahim and Rekasius criterion is identical to (5-25) in this case.

Ibrahim and Rekasius set $k_1 = k_2 = 6$ and $\beta_1 = \beta_2 = \frac{1}{6}$ getting

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{2}\frac{\frac{1}{2}\omega^2-1+\frac{5}{6}j\omega}{(j\omega+3)(j\omega+2)(5-j\omega)} \\ \\ \frac{1}{2}(\quad)^* & \frac{1}{6} \end{bmatrix} > 0$$

Applying Sylvester's conditions to this matrix gives

$$\frac{1}{36} - \frac{(1-\omega^2/2)^2 + 25\omega^2/36}{4(\omega^2+4)(\omega^2+9)(\omega^2+25)} > 0$$

or

$$1 - \frac{(3 - \frac{3}{2}\omega^2)^2 + \frac{25}{4}\omega^2}{(\omega^2+4)(\omega^2+9)(\omega^2+25)} > 0$$

This inequality is true for all ω so that the given system is absolutely stable for nonlinearities in the sector $[0, 6]$.

Instead of picking the β_i, they can be calculated by setting the coefficients of the $j\omega^3$ and ω^2 terms to zero. This gives $\beta_1 = \beta_2$ and $2 - 9\beta_1 = 0$ or $\beta_1 = 2/9$. The matrix is

$$\begin{bmatrix} \frac{1}{k_1} & \frac{1}{2}\,\frac{13j\omega/9 + 1}{(j\omega+3)(j\omega+2)(5-j\omega)} \\ \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} \end{bmatrix} > 0$$

Applying Sylvester's conditions gives

$$\frac{1}{k_1k_2} - \frac{169\omega^2/81 + 1}{4(\omega^2+4)(\omega^2+9)(\omega^2+25)} > 0$$

The maximum value of the frequency dependent term is approximately 1/1290. Therefore, if $k_1k_2 < 1290$ or $k_1 = k_2 < 35.8$, then the system is absolutely stable. This is a considerable improvement over the results of Ibrahim and Rekasius. However, it leaves room for improvement as a simple check shows that the linearized system is stable for all positive gain.

Example 5-3

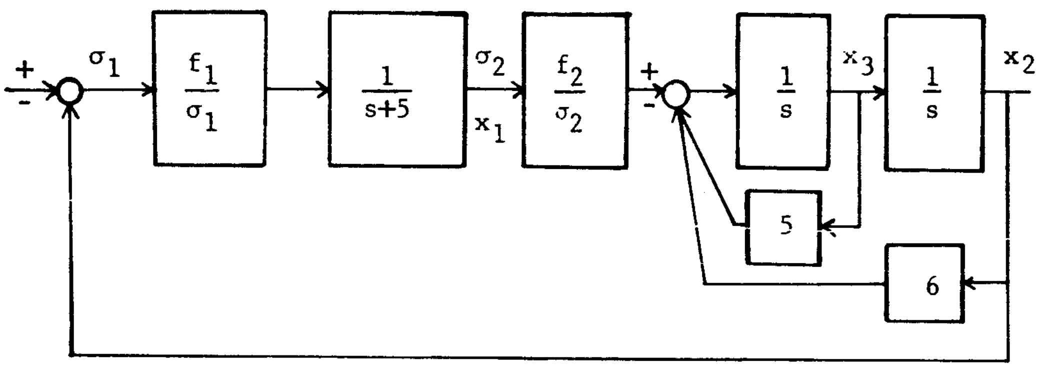
Consider now the same system as in Example 5-2 except that the s + 1 term in the numerator is missing. The block diagram and equations are given in Fig. 8. The term $\bar{\beta}C'B$ in this case is indefinite, and the theorem of Ibrahim and Rekasius cannot be applied. The term $C'A_\omega^{-1}B$ is

$$C'A_\omega^{-1}B = \begin{bmatrix} 0 & \frac{1}{(j\omega+3)(j\omega+2)} \\ \frac{-1}{(j\omega+5)} & 0 \end{bmatrix}$$

$$(j\omega\beta-I)C'A_\omega^{-1}B = \begin{bmatrix} 0 & \frac{(j\omega\beta_1-1)}{(j\omega+2)(j\omega+3)} \\ \frac{-j\omega\beta_2-1}{j\omega+5} & 0 \end{bmatrix}$$

Applying (5-25) gives the following as the stability criterion.

$$\begin{bmatrix} \frac{1}{k_1} & \frac{1}{2}\left[\frac{j\omega\beta_1-1}{(j\omega+2)(j\omega+3)} + \frac{1+j\omega\beta_2}{5-j\omega}\right] \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} \end{bmatrix} > 0$$



a) Block Diagram Defining the State Variables

$$\dot{\underline{x}} = A\underline{x} - B\underline{f}(\underline{\sigma}), \quad \underline{\sigma} = C'\underline{x}$$

$$A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad C' = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

b) System Equations

Fig. 8. System of Example 5-3.

This inequality can be rewritten as

$$\begin{bmatrix} \frac{1}{k_1} & \frac{1}{2}\left[\frac{1+(\beta_1-1-5\beta_2)\omega^2+(6+6\beta_2+5\beta_1)j\omega-\beta_2 j\omega^3}{(j\omega+2)(j\omega+3)(5-j\omega)}\right] \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} \end{bmatrix} > 0$$

Again setting the coefficients of $j\omega^3$ and ω^2 to zero gives $\beta_2 = 0$, $\beta_1 = 1$, and the resulting matrix is

$$\begin{bmatrix} \frac{1}{k_1} & \frac{1}{2}\left[\frac{11j\omega+1}{(j\omega+2)(j\omega+3)(5-j\omega)}\right] \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} \end{bmatrix} > 0$$

Applying Sylvester's conditions gives

$$\frac{1}{k_1 k_2} - \frac{(121\omega^2 + 1)}{4(\omega^2+4)(\omega^2+9)(\omega^2+25)} > 0.$$

The maximum magnitude of the second term is approximately 1/24.8 so that the system is absolutely stable for $k_1k_2 < 24.8$. A check on the linearized system shows that k_1k_2 must be less than 280 for stability.

5.3 The Time-Varying Case

Once again sufficient conditions for the time-varying case can be obtained by just considering the quadratic Liapunov function. The derivation of the previous section goes through with $\beta = 0$, and therefore, for the system

$$\dot{\underline{x}} = A\underline{x} - B\underline{f}(\underline{\sigma}, t)$$
$$\underline{\sigma} = C'\underline{x} \tag{5-31}$$

the condition for absolute stability is

$$K^{-1} + \mathrm{He}\, C' A_{\omega}^{-1} B > 0 \tag{5-32}$$

The case where the integrals of the time-varying nonlinear functions are retained in the Liapunov function is also amenable to treatment in a similar manner as in Chapter 4. The Liapunov function is

$$V = \underline{x}'P\underline{x} + \int_0^{\underline{\sigma}} \underline{f}'(\underline{z}, t)\bar{\beta}d\underline{z} \tag{5-33}$$

For this case $-\dot{V}$ is nothing more than (5-5) plus the time derivative of the integral term, or

$$-\dot{V} = \underline{x}'Q\underline{x} + \underline{f}(\underline{\sigma}, t)'(2B'P - \bar{\beta}C'A - C')\underline{x}$$
$$+ \underline{f}(\underline{\sigma}, t)'(\bar{\beta}C'B + K^{-1})\underline{f}(\underline{\sigma}, t) + \lambda(\underline{\sigma}, t)$$
$$- \int_0^{\underline{\sigma}} \frac{\partial \underline{f}'(\underline{z}, t)}{\partial t} \bar{\beta}\, d\underline{z} \tag{5-34}$$

The term $\lambda(\underline{\sigma}, t) = \underline{f}(\underline{\sigma}, t)'(\underline{\sigma} - K^{-1}\underline{f}(\underline{\sigma}, t)) \geq 0$ has been added and subtracted.

The three cases are considered again so that the inequalities are

Case I
$$\int_0^{\sigma_i} \frac{\partial f_i(z_i,t)}{\partial t}\, dz_i \leq \alpha_i^{I}\, \sigma_i^2 \qquad (5\text{-}35)$$

Case II
$$\int_0^{\sigma_i} \frac{\partial f_i(z_i,t)}{\partial t}\, dz_i \leq \alpha_i^{II} \sigma_i f_i(\sigma_i,t) \qquad (5\text{-}36)$$

Case III
$$\int_0^{\sigma_i} \frac{\partial f_i(z_i,t)}{\partial t}\, dz_i \leq \alpha_i^{III} f_i(\sigma_i,t)^2 \qquad (5\text{-}37)$$

In each case the summation of β_i multiplied by the right hand side of the inequality is added to and subtracted from $\dot{V}$ giving the following results.

Case I
$$\begin{aligned} -\dot{V} &= \underline{x}'(Q - C\overline{\beta\alpha}C')\underline{x} \\ &+ \underline{f}(\underline{\sigma},t)'\,(2B'P - \overline{\beta}C'A - C')\underline{x} \\ &+ \underline{f}(\underline{\sigma},t)'(\overline{\beta}C'B + K^{-1})\,\underline{f}(\underline{\sigma},t) + \lambda(\underline{\sigma},t) \\ &+ (\underline{\sigma}'\overline{\beta\alpha}\underline{\sigma} - \int_0^{\underline{\sigma}} \frac{\partial \underline{f}(\underline{z},t)'}{\partial t}\,\overline{\beta}\, d\underline{z}) \end{aligned} \qquad (5\text{-}38)$$

Case II $$-\dot{V} = \underline{x}'Q\underline{x} + \underline{f}(\underline{\sigma},t)'(2B'P - \bar{\beta}C'A - C' - \overline{\beta\alpha}C')\underline{x}$$

$$+ \underline{f}(\underline{\sigma},t)'(\bar{\beta}C'B + K^{-1})\underline{f}(\underline{\sigma},t) + \lambda(\underline{\sigma},t)$$

$$+ (\underline{\sigma}'\overline{\beta\alpha}\ \underline{f}(\underline{\sigma},t) - \int_0^{\sigma} \frac{\partial \underline{f}(\underline{z},t)'}{\partial t}\bar{\beta}\, d\underline{z}) \qquad (5\text{-}39)$$

Case III $$-\dot{V} = \underline{x}'Q\underline{x} + \underline{f}(\underline{\sigma},t)(2\beta'P - \bar{\beta}C'A - C')\underline{x}$$

$$+ \underline{f}(\underline{\sigma},t)'(\bar{\beta}C'B + K^{-1} - \overline{\beta\alpha})\underline{f}(\underline{\sigma},t) + \lambda(\underline{\sigma},t)$$

$$+ (\underline{f}(\sigma,t)'\ \overline{\beta\alpha}\ \underline{f}(\underline{\sigma},t) - \int_0^{\sigma} \frac{\partial \underline{f}(\underline{z},t)'}{\partial t}\bar{\beta}\, d\underline{z} \qquad (5\text{-}40)$$

where $\overline{\beta\alpha} = \text{diag}\ (\beta_1\alpha_1, \beta_2\alpha_2, \ldots, \beta_m\alpha_m)$. The last term in all these cases is positive.

Similar to the previous development, the expression for $-\dot{V}$ should be of the form

$$-\dot{V} = (Q_2'\underline{x} + T\underline{f}(\underline{\sigma},t))'(Q_2'\underline{x}+T\underline{f}(\underline{\sigma},t)) + \underline{x}'\epsilon D\underline{x} + \lambda(\underline{\sigma},t)$$
$$+ \text{(Positive term)} \qquad (5\text{-}41)$$

Comparing (5-41) with (5-38), (5-39) and (5-40) leads to the following set of equations.

Case I $$Q - C\overline{\beta\alpha}C' = Q_2Q_2' + \epsilon D$$
$$2Q_2T = 2PB - A'C\bar{\beta} - C \qquad (5\text{-}42)$$
$$T'T = \frac{1}{2}(\bar{\beta}C'B + B'C\bar{\beta}) + K^{-1}$$

Case II $$Q = Q_2 Q_2' + \epsilon D$$

$$2Q_2 T = 2PB - A' C\bar{\beta} - C - \overline{\beta\alpha} C \tag{5-43}$$

$$T'T = \frac{1}{2}(\bar{\beta} C' B + B' C \bar{\beta}) + K^{-1}$$

Case III $$Q = Q_2 Q_2' + \epsilon D$$

$$2Q_2 T = 2PB - A' C\bar{\beta} - C \tag{5-44}$$

$$T'T = \frac{1}{2}(\bar{\beta} C' B + B' C \bar{\beta}) + K^{-1} - \overline{\beta\alpha}$$

Applying the Lemma 2 to each case results in the following stability criteria.

Case I $$K^{-1} + \mathrm{He}(j\omega\bar{\beta}+I) C' A_\omega^{-1} B - B' A_\omega^{-1*} C \overline{\beta\alpha} C' A_\omega^{-1} B > 0 \tag{5-45}$$

Case II $$K^{-1} + \mathrm{He}(j\omega\bar{\beta}+I+\overline{\beta\alpha}) C' A_\omega^{-1} B > 0 \tag{5-46}$$

Case III $$K^{-1} - \overline{\beta\alpha} + \mathrm{He}(j\omega\bar{\beta}+I) C' A_\omega^{-1} B > 0 \tag{5-47}$$

When there is one time-varying element, these criteria reduce to the criteria of Rekasius and Rowland (1965), which are given in Chapter 4.

Example 5-4

The same example as in Example 5-2 is worked except that now the nonlinearities are assumed to be time-varying. The system equations are given in Fig. 7. The stability criterion given by (5-32) is illustrated first.

$$K^{-1} + \mathrm{He}\,C'A_{\omega}^{-1}B > 0 \tag{5-32}$$

$$C'A_{\omega}^{-1}B = \begin{bmatrix} 0 & \dfrac{1+j\omega}{(j\omega+3)(j\omega+2)} \\ & \\ \dfrac{-1}{j\omega+5} & 0 \end{bmatrix}$$

$$K^{-1} + \mathrm{He}\,C'A_{\omega}^{-1}B = \begin{bmatrix} \dfrac{1}{k_1} & \dfrac{1}{2}\left[\dfrac{1+j\omega}{(j\omega+3)(j\omega+2)} - \dfrac{1}{5-j\omega}\right] \\ & \\ \dfrac{1}{2}\left(\quad\right)^{*} & \dfrac{1}{k_2} \end{bmatrix}$$

Applying Sylvester's condition gives

$$\frac{1}{k_1k_2} - \frac{(2\omega^2-1) + \omega^2}{4(\omega^2+25)(\omega^2+4)(\omega^2+9)} > 0$$

The maximum magnitude of the frequency dependent terms is approximately 1/80 so that $k_1k_2 < 80$ is sufficient to insure that the system is stable.

To illustrate the second type of stability criterion, the time-varying nonlinearities are assumed to have specific forms. Let

$$f_1(\sigma_1,t) = (1 - \cos t)\ g_1(\sigma_1)$$

$$f_2(\sigma_2,t) = (1 - \cos 2t)\ g_2(\sigma_2)$$

where $0 \leq g_i(\sigma_i)/\sigma_i \leq k_i/2$. The partial derivatives with respect to time are

$$\frac{\partial f_1}{\partial t} = \sin t \; g_1(\sigma_1)$$

$$\frac{\partial f_2}{\partial t} = 2 \sin 2t \; g_2(\sigma_2)$$

Using the first type of constraint gives

$$\int_0^{\sigma_1} \sin t g_1(z_1) dz_1 \leq \frac{k_1 \sigma_1^2}{4} = \alpha_1 \sigma_1^2$$

$$\int_0^{\sigma_2} 2 \sin 2t g_2(z_2) dz_2 \leq \frac{k_2 \sigma_2^2}{2} = \alpha_2 \sigma_2^2$$

The stability criterion is given by (5-45) and is

$$K^{-1} + He(j\omega\bar{\beta}+I)C'A_\omega^{-1}B - B'A_\omega^{-1*} C\overline{\beta\alpha}C'A_\omega^{-1}B > 0 \tag{5-45}$$

The term $K^{-1} + He(j\omega\bar{\beta}+I)C'A_\omega^{-1}B$ has been calculated in Example 5-2. The term $B'A_\omega^{-1*}C\overline{\beta\alpha}C'A_\omega^{-1}B$ can be calculated giving

$$B'A_\omega^{-1*}C\overline{\beta\alpha}C'A_\omega^{-1}B = \begin{bmatrix} \dfrac{\beta_2 k_2}{2(\omega^2+25)} & 0 \\ & \\ 0 & \dfrac{\beta_1 k_1(\omega^2+1)}{4(\omega^2+4)(\omega^2+9)} \end{bmatrix}$$

Using the results of Example 5-2, that is $\beta_1 = \beta_2 = 2/9$, gives the stability criterion as

$$\begin{bmatrix} \frac{1}{k_1} - \frac{k_2}{9(\omega^2+25)} & \frac{1}{2}\left(\frac{(13\,j\omega/9 - 1)}{(j\omega+3)(j\omega+2)(5-j\omega)}\right) \\ \frac{1}{2}(\quad)^* & \frac{1}{k_2} - \frac{k_1(\omega^2+1)}{18(\omega^2+4)(\omega^2+9)} \end{bmatrix} > 0$$

Applying Sylvester's criterion to this matrix gives

$$\frac{1}{k_1} - \frac{k_2}{9(\omega^2+25)} > 0$$

and

$$\frac{1}{k_1k_2} - \frac{1}{9(\omega^2+25)} - \frac{(\omega^2+1)}{18(\omega^2+4)(\omega^2+9)}$$

$$+ \frac{k_1k_2(\omega^2+1)}{162(\omega^2+4)(\omega^2+9)(\omega^2+25)} - \frac{169\omega^2/81 + 1}{4(\omega^2+9)(\omega^2+4)(\omega^2+25)} > 0$$

In the first inequality the frequency dependent term is largest when $\omega^2 = 0$ so that

$$\frac{1}{k_1} - \frac{k_2}{9(25)} > 0$$

$$k_1k_2 < 225$$

In the second inequality the negative term which has the largest magnitude is $1/9(\omega^2+25)$. The frequency at which the magnitude is largest is $\omega^2 = 0$. It can be shown that the frequency at which the sum of the negative terms has its largest magnitude is also zero so that, if the inequality is satisfied at zero, it is satisfied for all ω. Letting $\omega^2 = 0$ gives

$$\frac{1}{k_1k_2} - \frac{1}{225} - \frac{1}{648} + \frac{k_1k_2}{162(4)(9)(25)} - \frac{1}{4(9)(4)(25)} > 0$$

Setting this equal to zero gives

$$(k_1k_2)^2 - (903.5)k_1k_2 + 14.6 \times 10^4 = 0$$

and solving this gives $k_1k_2 < 216.7$. Therefore, the new criterion gives a substantial improvement over the previous case since, in that case, $k_1k_2 < 80$ was the best that could be done. The criteria (5-46) and (5-47) cannot be used with the assumed nonlinearities because the values of α_1 are infinite.

5.4 The Particular Case

It does not appear that the general case of m nonlinearities with a zero eigenvalue in the A matrix can be handled by these methods. Of course the system equations must be manipulated so as to remove the equation which gives the zero eigenvalue, getting a matrix A_1, of order n-1 by n-1, and an additional $\dot{\xi}$ equation, as illustrated in Chapter 2, (2-5). A simple example is worked which shows that a Liapunov function of the proper form does not exist for a particular problem. Therefore, a general theory does not exist, since if it did, it could solve that problem.

The system and system equations are given in Fig. 9. The most general quadratic form of the two variables plus the integrals of the two nonlinearities is used as the Liapunov function.

$$V = \frac{a}{2} x^2 + b\xi x + \frac{\xi^2}{2} + \beta_1 \int_0^{\xi} f(z)dz + \beta_2 \int_0^{x} g(z)dz$$

The time derivative is

$$\dot{V} = -ax^2 - b\xi x - (\beta_2 - b)\, xg(x) - b\xi f(\xi)$$

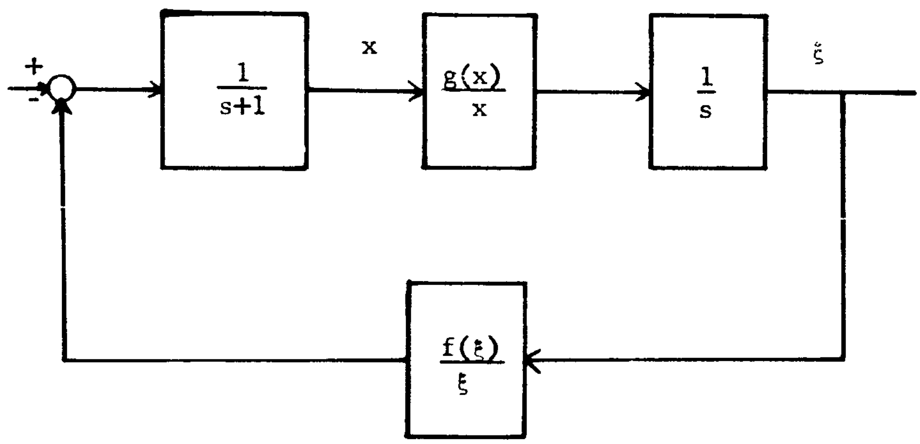
$$+ (\beta_1 - \beta_2) f(\xi) g(x) - axf(\xi)$$

The only definite term in ξ is $-b\xi f(\xi)$, but there is a term $b\xi x$ which is indefinite. Since $f(\xi)/\xi$ can have any value between 0 and k_1, the indefinite term can be positive and greater in magnitude than the negative definite term. However, since in the analogous one nonlinearity case there is no cross term in ξ in the Liapunov function, setting $b = 0$ removes this problem. However, if that is done in this case, then there is no definite term in ξ or $f(\xi)$ at all. Therefore, a Liapunov function of the proper form does not exist for this problem, since the most general quadratic plus integral form of Liapunov function was considered.

A Liapunov function which works for this simple system is

$$V = \int_0^{\xi} f(y)dy + \int_0^{x} g(z)dz$$

$$\dot{V} = -g(x)x$$



a) Block Diagram Defining the State Variables

$$\dot{\xi} = g(x), \qquad 0 \leq g(x)/x \leq k_2$$

$$\dot{x} = -x - f(\xi), \qquad 0 \leq f(\xi)/\xi \leq k_1$$

b) System Equations

Fig. 9. System with Two Nonlinearities and a Zero Eigenvalue.

Here $\dot{V}$ is semidefinite, but absolute stability is proven for time invariant nonlinearities. However, this Liapunov function cannot be generalized since, as before, this type of Liapunov function leads to an indefinite $\dot{V}$ in many cases.

There is a generalization of the simplest particular case which is amenable to treatment by the above methods. Letov (1961) considers this case for two nonlinearities in what he calls systems with two actuators. This would seem to be systems with motors, etc. operating in parallel. If m nonlinearities are considered, his equations can be generalized to be of the form

$$\begin{aligned} \dot{\underline{y}} &= A\underline{y} - B\underline{\xi} \\ \dot{\underline{\xi}} &= \underline{f}(\underline{\sigma}) \\ \underline{\sigma} &= C_1'\underline{y} - R_1\underline{\xi} \end{aligned} \tag{5-48}$$

where A has all its eigenvalues in the left half plane. This set of equations can be put into a form similar to (2-5) by using the transformation

$$\underline{x} = A\underline{y} - B\underline{\xi}$$

The equations become

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} - B\dot{\underline{\xi}} = A\underline{x} - B\underline{f}(\underline{\sigma}) \\ \dot{\underline{\xi}} &= \underline{f}(\underline{\sigma}) \\ \underline{\sigma} &= C_1'(A^{-1}\underline{x} + A^{-1}B\underline{\xi}) - R_1\underline{\xi} \\ \underline{\sigma} &= C'\underline{x} - R\underline{\xi} \end{aligned} \tag{5-49}$$

where these equations are now in a form which is analogous to (2-5). The matrix R is m by m while B and C are n by m. The Liapunov function is also analogous to V_1 of (3-8) and is

$$V = \underline{x}'P\underline{x} + \underline{\xi}'R'L R\underline{\xi} + \int_0^{\sigma} \underline{f}(\underline{z})'\bar{\beta}d\underline{z} \tag{5-50}$$

Taking $\dot{V}$ gives

$$\dot{V} = \underline{x}'(A'P + PA)\underline{x} - 2\underline{x}'PB\underline{f}(\underline{\sigma}) + \underline{\xi}'R'LR\dot{\underline{\xi}} + \dot{\underline{\xi}}'R'LR\underline{\xi}$$

$$+ \underline{f}(\underline{\sigma})'\bar{\beta}\dot{\underline{\sigma}} \tag{5-51}$$

Substituting for $\underline{\xi}'R'$, $\dot{\underline{\xi}}$ and $\dot{\underline{\sigma}}$ and collecting terms results in

$$\dot{V} = \underline{x}'(A'P + PA)\underline{x} - \underline{x}'(2PB - CL'R - CLR - A'C\bar{\beta})\underline{f}(\underline{\sigma})$$

$$-\underline{f}(\underline{\sigma})'(\bar{\beta}C'B + \bar{\beta}R')\underline{f}(\underline{\sigma}) - \underline{\sigma}'LR\underline{f}(\underline{\sigma}) - \underline{f}(\underline{\sigma})R'L\underline{\sigma} \tag{5-52}$$

Adding and subtracting $\lambda_1(\underline{\sigma}) = (\underline{\sigma} - K^{-1}\underline{f}(\underline{\sigma}))'LR\underline{f}(\underline{\sigma})$ and $\lambda_2(\underline{\sigma}) = \underline{f}(\underline{\sigma})'R'L(\underline{\sigma} - K^{-1}\underline{f}(\underline{\sigma}))$ to (5-52) results in the equation

$$-\dot{V} = \underline{x}'Q\underline{x} + x'(2PB - CL'R - CLR - A'C\bar{\beta})\underline{f}(\underline{\sigma})$$

$$+ \underline{f}(\underline{\sigma})'(\bar{\beta}C'B + \bar{\beta}R + K^{-1}LR + R'LK^{-1})\underline{f}(\underline{\sigma}) + \lambda_1(\underline{\sigma}) + \lambda_2(\underline{\sigma}) \tag{5-53}$$

The term $\underline{f}'(\bar{\beta}C'B + \bar{\beta}R + K^{-1} LR + R'LK^{-1})\underline{f}$ should be positive definite, and therefore the symmetric part of the matrix must also be positive definite. Since $-\dot{V}$ should be in the form given by (5-6), the following equations result.

$$Q = Q\,Q' + \epsilon\,D \tag{5-54}$$

$$2Q_2T = 2PB - CL'R - CLR - A'C\bar{\beta} \tag{5-55}$$

$$T'T = \mathrm{He}(\bar{\beta}C'B + \bar{\beta}R + K^{-1}LR + R'LK^{-1}) \tag{5-56}$$

The use of the **Hermitian** part of the matrix in (5-56) comes from the fact that the Hermitian part is the symmetric part for real matrices.

The necessary and sufficient conditions for the solution of equations (5-54) and (5-55) to exist as matrices $P > 0$ and Q_2 are that

$$T'T + 2\mathrm{He}H'A_\omega^{-1}G \; > \; 0 \qquad (**)$$

where

$$H = \frac{1}{2}CL'R + \frac{1}{2}CLR + \frac{1}{2}A'C\bar{\beta}$$

$$G = B$$

Substituting G, H and $T'T$ into (**) results in the inequality

$$\mathrm{He}(\bar{\beta}R + K^{-1}LR + R'LK^{-1})$$

$$+ \mathrm{He}(R'L + R'L' + j\omega\bar{\beta})C'A_\omega^{-1}\,B > 0 \tag{5-57}$$

Using the fact that $\mathrm{He}(R'L + R'L' + j\omega\bar{\beta})R/j\omega = \mathrm{He}\bar{\beta}R$ gives the result that the system (5-49) is absolutely stable if matrices $\bar{\beta}$ and L can be found such that the following matrix inequality holds for all real ω.

$$\mathrm{He}(K^{-1}LR + R'LK^{-1})$$

$$+ \mathrm{He}(R'L + R'L' + j\omega\bar{\beta})(C'A_{\omega}^{-1}B + R/j\omega) > 0 \qquad (5\text{-}58)$$

Consider the special case where R is a symmetric matrix. Let $2LR = I$. The stability criterion (5-58) becomes

$$K^{-1} + \mathrm{He}(I + j\omega\bar{\beta})(C'A_{\omega}^{-1}B + R/j\omega) > 0 \qquad (5\text{-}59)$$

For the case of one nonlinear element, (5-59) reduces to the criterion for the simplest particular case given in Chapter 3. By putting $2LR = I$ in (5-50), it is seen that R must be positive definite if V is to be definite. This is analogous to $\gamma > 0$ in the single nonlinearity case of Chapter 3.

<u>Example 5-5</u>

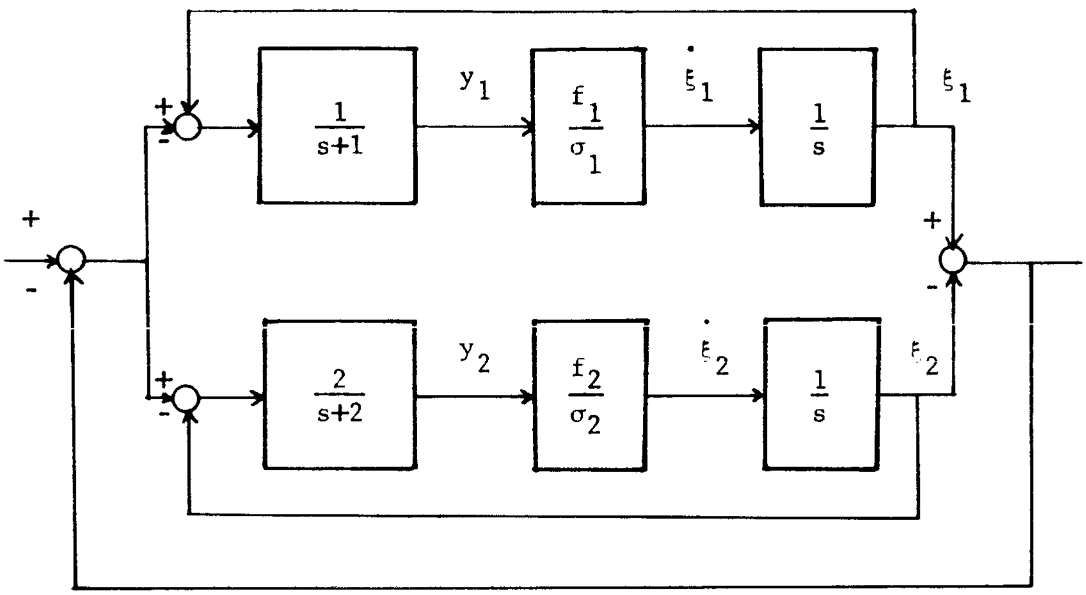
Consider the system given by the block diagram in Fig. 10. The system equations are

$$\dot{\underline{y}} = A\underline{y} - B\underline{\xi}$$

$$\dot{\underline{\xi}} = \underline{f}(\underline{y})$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$$



a) Block Diagram Defining the State Variables

$$y_1 = -y_1 - 2\xi_1 - \xi_2$$

$$y_2 = -y_2 - 2\xi_1 - 4\xi_2$$

$$\xi_1 = f(y_1)$$

$$\xi_2 = f(y_2)$$

b) System Equations

Fig. 10. System of Example 5-5.

Making the transformation $\underline{x} = A\underline{y} - B\underline{\xi}$ gives

$$\dot{\underline{x}} = A\underline{x} - B\underline{f}(\underline{\sigma})$$

$$\dot{\underline{\xi}} = \underline{f}(\underline{\sigma})$$

$$\underline{\sigma} = \underline{y} = A^{-1}\underline{x} + A^{-1}B\underline{\xi} = C'\underline{x} - R\underline{\xi}$$

where

$$C' = A^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \quad , \quad -A^{-1}B = R = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The stability criterion is

$$K^{-1} + He(I + j\omega\beta)(C'A_\omega^{-1}B + \frac{1}{j\omega} R) > 0 \qquad (5\text{-}59)$$

The term $C'A_\omega^{-1}B$ is

$$C'A_\omega^{-1}B = \begin{bmatrix} \frac{-2}{j\omega+1} & \frac{-1}{j\omega+1} \\ \frac{-1}{j\omega+2} & \frac{-2}{j\omega+2} \end{bmatrix}$$

$$C'A_\omega^{-1}B + \frac{1}{j\omega}R = \begin{bmatrix} \frac{2}{j\omega(j\omega+1)} & \frac{1}{j\omega(j\omega+1)} \\ \frac{2}{j\omega(j\omega+2} & \frac{4}{j\omega(j\omega+2)} \end{bmatrix}$$

$$(I + j\omega\bar{\beta})(C'A_\omega^{-1}B + \frac{1}{j\omega}R) = \begin{bmatrix} \frac{2(1+j\omega\beta_1)}{j\omega(j\omega+1)} & \frac{1+j\omega\beta_1}{j\omega(j\omega+1)} \\ \\ \frac{2(1+j\omega\beta_2)}{j\omega(j\omega+2)} & \frac{4(1+j\omega\beta_2)}{j\omega(j\omega+2)} \end{bmatrix}$$

Taking the Hermitian part of this gives

$$\mathrm{He}(I+j\omega\bar{\beta})(C'A_\omega^{-1}B + \frac{1}{j\omega}R) = \frac{1}{2}\begin{bmatrix} \frac{2(2\beta_1-2)}{\omega^2+1} & (\quad)^* \\ \\ \frac{(\beta_1-2\beta_2)j\omega+(2\beta_2+2\beta_1-3)}{(j\omega+2)(1-j\omega)} & \frac{4(4\beta_2-2)}{\omega^2+4} \end{bmatrix}$$

If $\beta_1 = 2\beta_2 = 1$, this becomes

$$\mathrm{He}(I + j\omega\bar{\beta})(C'A_\omega^{-1}B + \frac{1}{j\omega}R) = 0$$

Therefore, the stability criterion is

$$K^{-1} > 0$$

so that the system is stable for all positive k_i.

The above results say that Q_2 and P must be zero. In a similar manner to the first example in section 5-2, this implies that $H = CLR + \frac{1}{2}A'C\bar{\beta} = 0$. The proper quantities are put in this equation to see if $P = 0$, $Q_2 = 0$ is really a solution of the set of equations (5-54) and (5-55).

The result is

$$CLR + \frac{1}{2}A'C\overline{\beta} = C + A'C\overline{\beta}$$

since 2LR = I.

$$\begin{bmatrix} -1 & 0 \\ & \\ 0 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ & \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ & \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ & \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ & \\ 0 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ & \\ 0 & \frac{1}{2} \end{bmatrix} = 0$$

Therefore H = 0 is satisfied so that P = 0, Q_2 = 0 is a solution to the set of equations. This should be compared with the results obtained in Example 5-2.

5.5 Conclusions

In this chapter frequency domain stability criteria are obtained for systems with more than one nonlinear and/or time-varying element. This is accomplished by first proving Lemma 2, which is a generalization of Lemma 1 of Chapter 3. Lemma 2 states that a matrix, which is a function of frequency, must be positive definite for all frequency if a set of algebraic equations is to have a solution. In the case where the matrix is one by one, Lemma 2 reduces to Lemma 1.

Lemma 2 is used in conjunction with the Second Method of Liapunov to obtain stability criteria for the principal case of systems with more than one nonlinear and/or time-varying element. In the case where there is only one nonlinear and/or time-varying element, these criteria reduce to the criteria given in Chapters 3 and 4. The criterion for the time-invariant nonlinear case is shown to be equivalent to the criterion obtained by Popov (1960) and better than the criterion obtained by Ibrahim and Rekasius (1964). The criterion for the time-varying case, which extends the work of Rekasius and Rowland (1965), has not been obtained previously.

The particular case of systems with more than one nonlinearity is discussed, and a stability criterion is given for a special particular case. Again, if there is only one nonlinearity, this case reduces to the simplest particular case of Chapter 3. The general particular case does not appear to be manageable by the methods of this chapter.

This completes the development of stability criteria for systems with more than one nonlinear and/or time-varying element. The next chapter contains concluding statements and indicates areas of further work on this subject matter.

Chapter 6

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

6.1 Conclusions

The absolute stability of nonlinear and time-varying systems is studied by use of a Liapunov function made up of a quadratic form plus the sum of the integrals of the nonlinear terms. The frequency criterion of Popov is extended by extending the matrix-inequality method of Yakubovich. This criterion is shown to be necessary and sufficient for the existence of the Lurie type Liapunov functions, and it is generally easier to use than the Liapunov approach. Extensions to time varying systems are given.

The objection can be raised that the results are not too good since it was shown that for certain problems there are other V-functions which give better results, and also that the results do not come close to the linearized system stability region in many cases. The answer to this objection is essentially that this is the best that can be done at this stage of the development of the theory. There is no way available of finding the best Liapunov function for a given system. The method developed in this work allows for a logical process of determining stability. If the frequency condition is met, it guarantees the existence of a positive definite V and $-\dot{V}$.

In trying to find Liapunov functions for high order systems, one of the main difficulties is that there is no easy way of testing high order non-quadratic or partially quadratic forms for positive definiteness. Therefore, although maybe not giving the best Liapunov function, the methods presented here give some results which may be adequate for a given problem, and which also may be the only results which can be obtained in a reasonably simple manner.

In summary the main contributions of this work are:

1. The matrix inequality method is extended by means of proving Lemma 2. This results in an extension of the Popov stability criterion from the scalar to the matrix case.
2. By using Lemma 2, the work of Rekasius and Rowland for time-varying system is extended to systems with many time-varying elements.

Other contributions of this work are:

1. By using the extended frequency criteria, the work of Yakubovich (1946c) on forced systems is extended to systems with many non-linearities. This is given in the Appendix.
2. Some indication is given as to when the criteria of Rekasius and Rowland can be used to get improved results.
3. The work of Bongiorno, Sandberg, and Narendra and Goldwyn is compared with the Popov criterion and is shown to be equivalent to it.
4. The criteria of Ibrahim and Rekasius and of Popov are compared with the criterion which is derived in this work for the case of many nonlinearities.

6.2 Further Work

There is a good deal of room for improvement and extensions of the results which have been presented here, since only sufficient conditions for stability are given by the Second Method. One way of improving the sufficient conditions would be by getting some information as to the slope of the nonlinearity into the Liapunov functions. Absolute stability means that the system must be stable for any nonlinearity in the sector, no matter how violent the changes in its slope are. By incorporating some constraints on the slope of the nonlinearity better results should be obtainable. Some results on this approach have been obtained by Brockett and Williams (1965) for the case of symmetric, monotonic nonlinearities.

Further investigations into forced systems should also prove fruitful, since most physical systems have some input forcing function. Also, digital computer programs to aid in the computational aspects of the problems can be investigated.

The cases of systems with zero and pure imaginary eigenvalues need further work, especially the cases of more than one nonlinear and/or time-varying element. The simplest particular case of the time-varying system does not have a stability criterion which is similar to the Rekasius and Rowland criteria. This also should be investigated.

Finally, there are results available for absolute stability by means of Popov's criterion for systems with time delay (Popov and Halanay 1962), systems with hysteresis and discontinuous nonlinearities

(Gelig 1964) and sampled-data systems (Jury and Lee 1964). Applications of Lemma 2 to these types of systems should lead to extensions of the existing results.

Appendix A

ABSOLUTE STABILITY OF FORCED SYSTEMS

The matrix-inequality method can be used also for forced systems. Again this is the work of Yakubovich (1964c). The system equation that he considers is

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} - \underline{b}f(\sigma) + \underline{r}(t) \\ \sigma &= c'\underline{x} \end{aligned} \tag{A-1}$$

where $\underline{r}(t)$ is a vector function bounded for $-\infty < t < \infty$. Yakubovich considers the case where $f(\sigma)$ is discontinuous so that he can take into account the possibility of a sliding regime. The results he obtained are not discussed in detail here, but it is shown that the previous extensions to m nonlinearities can also be made in this case.

The system equations for the more general case are

$$\begin{aligned} &\dot{\underline{x}} = A\underline{x} - B\underline{f}(\underline{\sigma}) + \underline{r}(t) \\ \underline{\sigma}' = C'\underline{x}, \qquad &0 \leq f_i(\sigma_i) \leq k_i\sigma_i^{\,2}, \qquad \underline{f}(\underline{0}) = \underline{0} \end{aligned} \tag{A-2}$$

The method is illustrated by proving the following theorem.

Theorem: In the system (A-2), let A have all its eigenvalues in the left half plane, let $\underline{r}(t)$ be bounded for $-\infty < t < \infty$ and let the condition

$$K^{-1} + \mathrm{He}\,C'A_{\omega}^{-1}B > 0 \tag{A-3}$$

be satisfied for all real ω.

Then

a) any solution of (A-2) is bounded for $t_0 \leq t < \infty$,

b) in the state space $\{x\}$ there is a bounded region F such that any solution reaches this region at some time, and for $t \geq t_0$ and $\underline{x}(t_0) \epsilon$ F it follows that $x(t) \epsilon$ F ,

c) there is a number $\mu > 0$ such that, for any two solutions $\underline{x}_1(t)$ and $\underline{x}_2(t)$ and $t \geq t_0$

$$|\underline{x}_1(t) - \underline{x}_2(t)| \leq \text{const. } e^{-\mu(t-t_0)} |\underline{x}_1(t_0) - \underline{x}_2(t_0)| \qquad \text{(A-4)}$$

The simple quadratic Liapunov function, $V = \underline{x}'P\underline{x}$ is used. Differentiating V gives

$$-\dot{V} = \underline{x}'Q\underline{x} + \underline{x}'(2PB - C)\underline{f}(\underline{\sigma}) + \underline{f}(\underline{\sigma})'K^{-1}\underline{f}(\underline{\sigma})$$

$$+ \lambda(\underline{\sigma}) - 2\underline{x}'P\underline{r}(t) \qquad \text{(A-5)}$$

which is just (5-5) with $\overline{P} = 0$ plus the term with $\underline{r}(t)$. If K^{-1} + $\text{HeC}'A_\omega^{-1}B > 0$, then $-\dot{V}$ can be written as

$$-\dot{V} = (Q_2'\underline{x} + T\underline{f}(\underline{\sigma}))'(Q_2'\underline{x} + T\underline{f}(\underline{\sigma}))$$

$$+ \underline{x}'D\underline{x} + \lambda(\underline{\sigma}) - 2\underline{x}'P\underline{r}(t) \qquad \text{(A-6)}$$

Since the first term is positive semidefinite and $\lambda(\underline{\sigma})$ is positive, $-\dot{V}$ can be rewritten as an inequality.

$$-\dot{V} \geq \underline{x}'D\underline{x} - 2\underline{x}'P\underline{r}(t) \qquad \text{(A-7)}$$

For any positive definite quadratic forms $\underline{x}'D\underline{x}$ and $\underline{x}'P\underline{x}$, there is a constant μ such that $\underline{x}'D\underline{x} \leq \mu\underline{x}'P\underline{x} = \mu V$. Therefore

$$\dot{V} \leq -\mu V + 2\underline{x}'P\underline{r}(t) \tag{A-8}$$

Since $\underline{r}(t)$ is bounded, the term $2\underline{x}'P\underline{r}(t)$ is going to be less than $\alpha\, V^{1/2}$ for some value of α. Therefore

$$\dot{V} \leq -\mu V + \alpha\, V^{1/2} \tag{A-9}$$

For some constant $V = C$, $-\mu C + \alpha\sqrt{C} = 0$ or $C = \frac{\alpha^2}{\mu^2}$. Therefore $V = \frac{\alpha^2}{\mu^2}$ defines an ellipsoid in the state space. For any solution starting outside the ellipsoid, $\dot{V}$ is negative and the solution eventually enters the ellipsoid. Any solution starting inside the ellipsoid must stay inside since $\dot{V}$ is negative outside the ellipsoid. This proves (a) and (b) of the theorem.

Writing $\underline{y} = \underline{x}_1 - \underline{x}_2$, $\underline{\sigma}_j = C'\underline{x}_j$, $\underline{\sigma}_0 = \underline{\sigma}_1 - \underline{\sigma}_2$ and $\underline{f}_0 = \underline{f}(\underline{\sigma}_1) - \underline{f}(\underline{\sigma}_2)$ leads to

$$\dot{\underline{y}} = \dot{\underline{x}}_1 - \dot{\underline{x}}_2$$

$$\dot{\underline{y}} = A\underline{x}_1 - B\underline{f}(\underline{\sigma}_1) + \underline{r}(t) - (A\underline{x}_2 - B\underline{f}(\sigma_2) + \underline{r}(t))$$

$$\dot{\underline{y}} = A(\underline{x}_1 - \underline{x}_2) - B(\underline{f}(\underline{\sigma}_1) - \underline{f}(\underline{\sigma}_2))$$

$$\dot{\underline{y}} = A\underline{y} - B\underline{f}_0 \tag{A-10}$$

Repeating the above calculations gives

$$-\dot{V} = \underline{y}'Q\underline{y} + \underline{y}'(2PB - C)\underline{f}_0 + \underline{f}_0'K^{-1}\underline{f}_0 + \lambda(\underline{\sigma}) \tag{A-11}$$

where K^{-1} comes from the additional condition that

$$0 \leq \frac{f_i(\sigma_{1i}) - f_i(\sigma_{2i})}{\sigma_{1i} - \sigma_{2i}} \leq k_i \quad .$$

(If $\underline{\sigma}_2 = 0$, this reduces to the previous inequality for the nonlinearity.) Therefore, as before,

$$\dot{V}(\underline{y}) \leq \mu V(\underline{y}) \tag{A-12}$$

This means that

$$V(\underline{y}(t)) \leq \text{const.} \; e^{-\mu(t-t_0)} \; V(\underline{y}(t_0)) \tag{A-13}$$

so that the magnitude of y is decreasing exponentially. Therefore

$$|\underline{x}_1(t) - \underline{x}_2(t)| \leq \text{const.} \; e^{-\mu(t-t_0)} |\underline{x}_1(t_0) - \underline{x}_2(t_0)| \tag{A-4}$$

and the proof is complete.

It is easily seen that this theorem holds true if the $\underline{f}(\underline{\sigma})$ in this case is also a function of time, i.e., $\underline{f}(\underline{\sigma},t)$. This is because V does not have the integral term in it, so that there is no change in the above proof if $\underline{f}(\underline{\sigma})$ is replaced by $\underline{f}(\underline{\sigma},t)$.

Appendix B

AN APPLICATION TO THE NUCLEAR ROCKET PROBLEM

This section contains an example in which the stability theory developed above is applied to the simplified nuclear rocket propulsion control system which was considered by Mohler (1962). This particular system is studied here because Mohler gives an analog computer diagram, which is used to obtain the system equations.

The stability of the operating point of this system can be found by linearizing the system equations. However, this procedure only gives stability information for some arbitrarily small region about the operating point. By treating the nonlinear and cross-coupling terms as time-varying coefficients, an attempt is made to obtain stability information in some finite region about the operating point.

The block diagram of the system is given in Figure 11. The desired thrust F_d is assumed to be constant and the actual thrust is given by the equation

$$F = c_1 \sqrt{T_c}\, \dot{W} \tag{B-1}$$

where c_1 is a constant, T_c is the propellant temperature at the core exit, and $\dot{W}$ is the propellant weight flow rate. The compensation and value and turbopump equations are

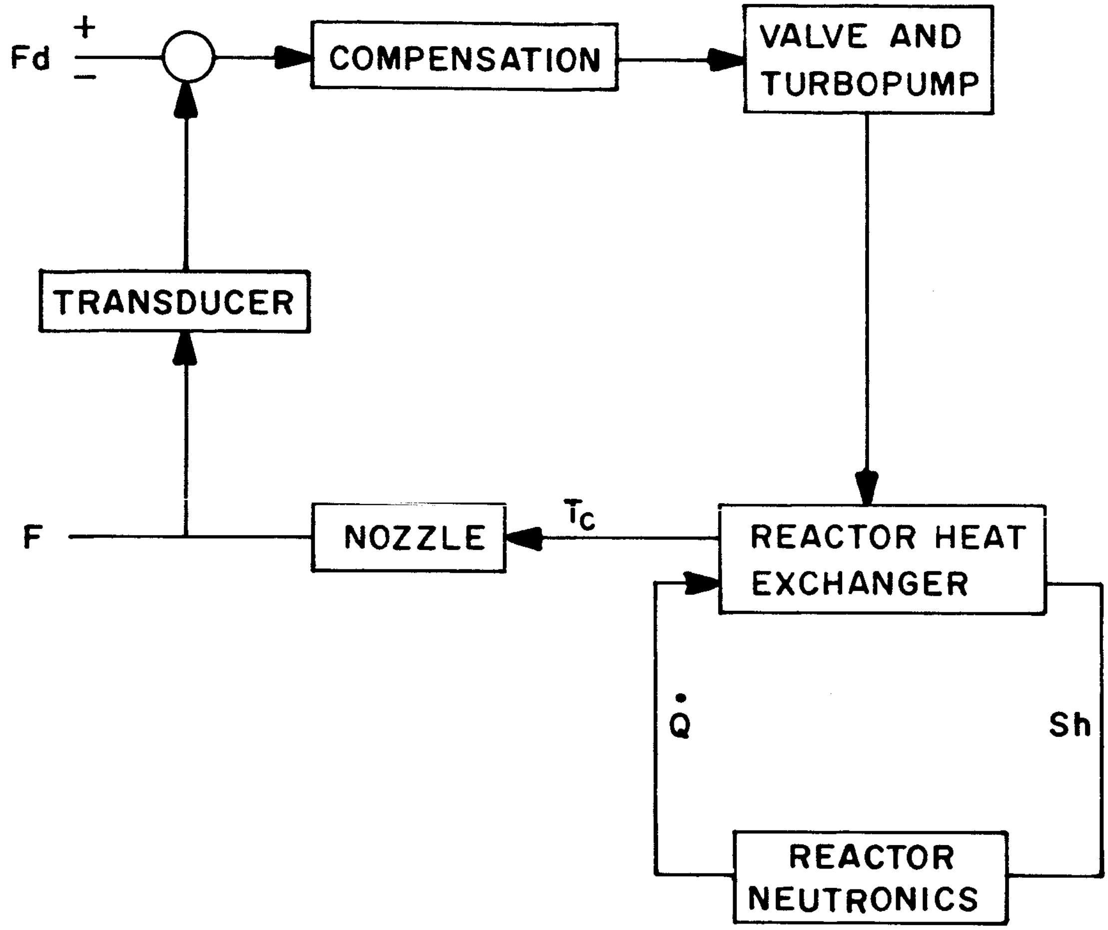


Fig. 11 Block Diagram of Simple Nuclear Rocket Propulsion Control System

$$\dot{y} = 0.1(F_d - F) \tag{B-2}$$

$$\ddot{W} = -2\dot{W} + .04\ (F_d - F) + y \tag{B-3}$$

where y is a state variable without physical significance.

The reactor heat exchanger equations are

$$\dot{Q} = c_2\dot{T}_f + c_3h(T_f - T_g) \tag{B-4}$$

$$c_3h(T_f - T_g) = c_4\dot{W}(T_c - T_i) \tag{B-5}$$

where c_3, c_4, and c_5 are constants, T_f is the average fuel moderator temperature, T_g is the average propellant temperature in the core, and h is the heat transfer coefficient due to convection, and T_i is the propellant temperature at the core entrance. T_g is approximated by

$$Tg \approx \frac{Ti + Tc}{2} \approx \frac{Tc}{2} \tag{B-6}$$

Mohler considered the neutron dynamics to be approximated by the average, one delayed neutron group approximation. The equations are

$$\ddot{Q} = 10^4(\delta k - .0065)\dot{Q} + .1C \tag{B-7}$$

$$\dot{C} = -.1C + \dot{Q} \tag{B-8}$$

where C is the concentration of delayed neutrons, $\dot{Q}$ is the reactor power, and δk is the reactivity. The reactivity is assumed to consist of three parts; rod reactivity, temperature reactivity, and propellant reactivity.

The above equations are now manipulated in such a manner that they can be written as

$$\underline{\dot{x}} = A\underline{x} - B\underline{f}(\underline{\sigma},t) \tag{B-9}$$

The state variables are y, $\dot{W}$, T_f, $\dot{Q}$, and C. Substituting for T_g in (B-5) and solving for T_c gives the equation

$$T_c = \frac{c_3 h T_f + T_c(c_4\dot{W} - c_3 h/2)}{c_4\dot{W} + c_3 h/2} \approx \frac{c_3 h T_f}{c_4\dot{W} + c_3 h/2} \qquad \text{(B-10)}$$

The heat transfer coefficient due to convection is

$$h \approx c_6 \dot{W}^{0.8}$$

Therefore

$$T_c \approx \frac{c_6 \dot{W}^{0.8} T_f}{c_4\dot{W} + c_3 c_6 \dot{W}^{0.8}/2} = f(\dot{W}) T_f \qquad \text{(B-11)}$$

The five equations are

$$\begin{aligned}
y &= 0.1\,(F_d - c_1\sqrt{f(\dot{W})T_f}\;\dot{W}) \\
\ddot{W} &= -2\dot{W} + .04\,(F_d - c_1\sqrt{f(\dot{W})T_f}\;\dot{W}) + y \\
\dot{T}_f &= \frac{1}{c_2}\dot{Q} - \frac{c_3 c_6}{c_2}\dot{W}^{0.8}\,(T_f - f(\dot{W})T_f/2) \\
\ddot{Q} &= 10^4\,(\delta k - .0065)\,\dot{Q} + .1C \\
\dot{C} &= -.1C + \dot{Q}
\end{aligned} \qquad \text{(B-12)}$$

The design conditions are

$$F_d = 10^8 \text{ lb}$$

$$\dot{W}_o = 1.2 \times 10^5 \text{ lb/sec}$$

$$\dot{Q}_o = 2.79 \times 10^9 \text{ BTU/sec}$$

$$T_{fo} = 4500^{\circ} \text{ R}$$

By making the transformation

$$x = y - 2.5\dot{W} \tag{B-13}$$

the first two equations are changed so that the nonlinear cross-coupling term appears only in one of the first two equations. The result, after substituting the coefficient values obtained from Mohler's computer diagram, is

$$\begin{aligned} \dot{x} &= -5\dot{W} - x \\ \ddot{W} &= .5\dot{W} + .04(F_d - 10.73\sqrt{f(\dot{W})T_f}\,\dot{W}) + x \\ \dot{T}_f &= \frac{\dot{Q}}{900} - .192\,T_f\,(1 - \tfrac{1}{2}f(\dot{W})\,\dot{W}^{0.8} \\ \ddot{Q} &= 10^4\,(k - .0065)\,\dot{Q} + .1C \\ \dot{C} &= -.1C + \dot{Q} \end{aligned} \tag{B-14}$$

where $f(\dot{W}) = 1/(.0232\dot{W}^{0.2} + 0.5)$.

These equations can now be put in the following form.

$$\begin{aligned} \dot{x} &= -5\dot{W} - x \\ \ddot{W} &- .5\dot{W} + x + a(t)\dot{W} \end{aligned} \tag{B-15}$$

where $a(t) = .04\,(F_d/\dot{W} - 10.73\sqrt{f(\dot{W})T_f}$

$$\dot{T}_f = \frac{\dot{Q}}{900} - b(t)\,T_f \tag{B-16}$$

where $b(t) = .192\,(1 - f(\dot{W})/2)\dot{W}^{0.8}$, and

$$\begin{aligned} \ddot{Q} &= 10^4\,(k(t) - .0065)\,\dot{Q} + .1C \\ \dot{C} &= -.1C + \dot{Q} \end{aligned} \tag{B-17}$$

The equations 13-15, B-16, and B-17 are three sets of uncoupled, linear, time-varying equations. The $\dot{Q}$ term in (B-10) acts as a forcing function and this equation is stable as long as b(t) remains positive. The stability criterion of Chapter 4 can be applied to (B-15) and (B-17).

Consider (B-15). At the operating point, $a(t) = 0$. Therefore, for changes in T_f and $\dot{W}$, $-k < a(t) < k$, and the Bongiorno type of stability criterion can be applied. In matrix notation (B-15) is

$$\begin{bmatrix} \dot{x} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ W \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} a(t)\dot{W} \qquad \text{(B-18)}$$

The transfer function is $G(j\omega) = \underline{c}' A_\omega^{-1} \underline{b}$.

$$G(j\omega) = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} j\omega - .5 & -5 \\ 1 & j\omega + 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(B-19)}$$

where $\Delta = (j\omega + 1)(j\omega - .5) = (j\omega)^2 + .5j\omega + 4.5$.

$$G(j\omega) = \frac{j\omega + 1}{(j\omega)^2 + .5j\omega + 4.5} \qquad \text{(B-20)}$$

The stability criterion is

$$k \left| G(j\omega) \right| < 1$$

The maximum value of $|G(j\omega)| = 2.2$ so that $k<1/2.2 = .45$ is sufficient for stability. To obtain some idea of what this means in terms of the state variables, assume that the flow rate increases suddenly, while T_f cannot change instantaneously. Then

$$a(t)_{min} = .04\left(\frac{Fd}{\dot{W} + \delta\dot{W}} \quad 10.73 \sqrt{f(\dot{W} + \delta\dot{W})T_f} \right) = -.45$$

and $\delta\dot{W}_{max}$ is approximately 1600 lb/sec. This corresponds to a change in thrust of about 13 X 10^5 lbs.

Equation (B-17) can be studied in the same manner as in Example 4-2, and similar information can be obtained.

This section has presented an approach to complicated nonlinear systems which allows some information to be determined about the stability region at the operating part of that system. As can be seen from the discussion of (B-15), the results are very conservative. However, since the original equations are nonlinear with cross-coupling, a suitable Liapunov function, which would give better results, is not known.

Appendix C

SOME CONSIDERATIONS IN THE PARALLEL ACTUATOR PROBLEM

This section contains a brief discussion of some of the qualitative aspects of operating devices, such as pumps, in parallel. The discussion requires using the theoretical concept of controllability and looking at some of its practical implications.

A system is completely controllable if every desired transition of the system's state can be effected in finite time by some unconstrained control inputs. Mathematically, this concept reduces to the linear indepencence of certain scalar or vector time functions. The mathematical details are not gone into here, but they are contained in the paper by Kreindler and Sarachik (1964).

The simplest example of a system which is not completely controllable is shown in Fig. 12, where the state $\underline{x}$ can be controlled only along (or parallel to) the line $x_1 = x_2$, rather than in the whole two-dimensional state space. Kreindler and Sarachik contend that this does not matter if one is only interested in the control of the output y. However, in the practical use there are definite limits on the values that x_1 and x_2 can attain. Therefore, if they are not controlled, then the output of the system may also become uncontrollable.

In the case of actuators such as pumps operating in parallel, similar problems exist. However, in the case of pumps, there is even the possibility of one pump getting to the point where the flow is actually going backwards through the pump. In that case a circulating flow is set up through the two pumps as shown in Fig. 13.

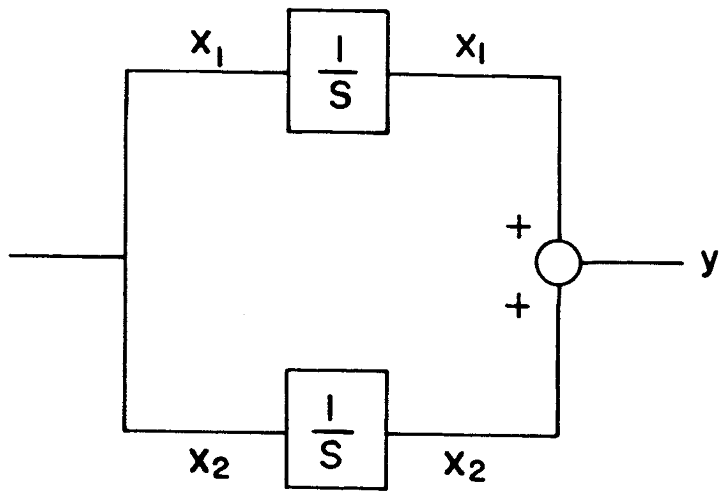


Fig. 12 System Which Is Not Completely State Controlled

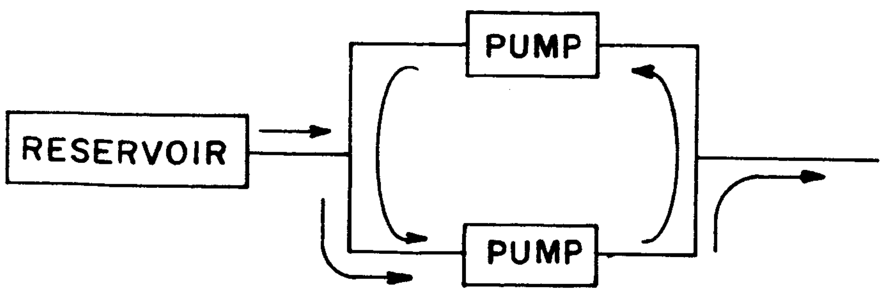


Fig. 13 Circulating Flow in Parallel Pump Operation

The solution to this problem is to feed back information as to the actual output and the output of each pump in such a manner as to keep both pumps pumping their share. Consider for example Fig. 14 as a possible configuration. The stability of the type of systems given in Fig. 14 is discussed in Chapter 5 with section 5-4 being especially pertinent.

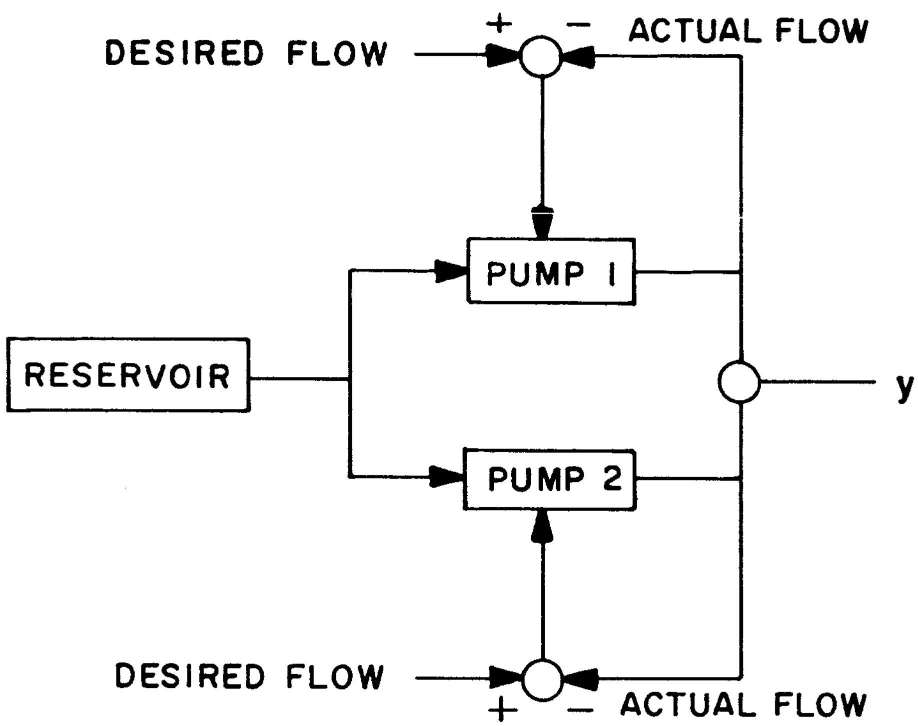


Fig. 14 Parallel Pumps with Feedback

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